1. Simple and Compound Interest

Simple Interest

- In simple interest there is one deposit and one repayment.

$$C_1 = C_0(1+i) \qquad C_0 = \frac{1}{1+i}C_1$$

- Over several time units, this becomes $C_n = C_0(1+ni)$.
- There are different conventions for the number of days in a year. ACT/365 assumes that there are 365 days.
- Calculations regarding simple interest must be done with respect to one fixed time point, called the focal point or valuation date.

Compound Interest

- Interest is given on top of existing interest. Assuming constant rate:

$$C_n = C_0(1+i)^n$$

- The discount factor is then $v = \frac{1}{1+i}$
- If an investment produces interest i_1, then the interest i_a adjusted for inflation i_0 is given by $i_a = \frac{i_1 - i_0}{1+i_0}$
- By definition, the nominal rate of interest compounded mthly, $i^{(m)}$, leads to a growth of $\left(1+\frac{i^{(m)}}{m}\right)^m$, in one time period.

e.g a $\underbrace{\$1000}_{C_0}$ loan for one year @ $\underbrace{15\%}_{i^{(m)}}$ p.a compounded $\underbrace{\text{monthly}}_{m=12}$

$$C_n = C_0\left(1+\frac{i^{(m)}}{m}\right)^{mn} \Rightarrow C_1 = C_0\left(1+\frac{i^{(m)}}{m}\right)^m$$

$$= 1000\left(1+\frac{0.15}{12}\right)^{12}$$

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- The effective interest is the effective growth in one time period (annual by default.
$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1$$
  - alternatively, if the period p is given ($p = \frac{1}{m}$), then $i_{ep} = p\,i^{(m)} = \frac{i^{(m)}}{m}$ is called the effective interest rate over period p.
  - effective rates for different time periods can be converted with:
$$(1 + i_{eq})^{1/q} = (1 + i_{ep})^{1/p}$$
  - nominal rate (period p) = frequency x effective rate (freq = $1/p$)
- Always assume given interest rates are compounded annually, unless the time period is less than one year, in which case you should assume simple interest.
- The accumulation factor is the accumulated value after one time period, i.e $A(p) = (1+i)^p = 1 + p\,i_p$
- In continuous compounding, we let $m \to \infty$:
$$i = \lim_{m\to\infty}\left(\left(1 + \frac{i^{(m)}}{m}\right)^m - 1\right) = e^{\delta} - 1$$
  - δ is the force of interest, i.e $\lim_{m\to\infty} i^{(m)}$
  - $c_n = c_0 e^{n\delta}$

The Discount rate

- The interest rate on a deposit can also be called the discount rate on the repayment, i.e PV x interest rate = FV x discount rate
- The discount rate is linked to the discount factor by $v = 1 - d$
- For a simple discount rate, $c_0 = (1-nd)c_n$ by definition. Thus:
$$(1+ni)(1-nd) = 1$$

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· Investment terminology:

- the coupon is an interest payment based on the face/nominal value of some instrument.
- the yield is the interest rate that can be earnt, and is implied by the market price relative to the maturity price.
- a discount instrument does not pay a coupon: simple interest is used for calculations because there is only one repayment.

· As with interest, one can have nominal or effective discount rates

$$1-d=\left(1-\frac{d^{(m)}}{m}\right)^m \Rightarrow \left(1+\frac{i^{(m)}}{m}\right)\left(1-\frac{d^{(m)}}{m}\right)=1$$

· If we borrow \$1 at $t=0$ for repayment at $t=1$ with annual interest i:

- repaying with equal instalments at the end of each m^{th} subinterval leads to a total interest payment of $i^{(m)}$
- repaying with equal instalments at the start of each m^{th} subinterval leads to a total interest payment of $d^{(m)}$

· For continuous compounding, $d_{ep}=1-e^{-p\delta}$.

Time dependent interest rates

· Suppose we have a case of continuous compounding where interest varies with time, i.e $i_p(t)$ gives the nominal interest rate for period p starting at time t. Then: $A(t+p)=A(t)[1+p\,i_p(t)]$

· In the limit as $p\to 0$ of $i_p(t)$, we find that:

$$\delta(t)=\lim_{p\to 0} i_p(t)=\lim_{p\to 0}\left[\frac{1}{A(t)}\,\frac{A(t+p)-A(t)}{p}\right]=\frac{A'(t)}{A(t)}.$$

Thus: $$A(t_2)=A(t_1)\exp\left(\int_{t_1}^{t_2}\delta(s)\,ds\right)$$

· The present value corresponding to the force of interest function is given by

$$v(t)=\frac{1}{A(t)}=\exp\left(-\int_0^t \delta(s)\,ds\right)$$

2. Cash flows, equations of value, and project appraisal

Cash flows

- Zero-coupon bonds provide a specified cash amount at a future time
- Fixed interest security: coupon is paid at $t=1,2,3,\dots,n$, and a lump sum is paid at $t=n$

time	0	1	...	n-1	n
cash flow	$-C_0$	c	...	c	$c+s$

- Indexed-link securities have coupons linked to some index
- Annuity: investor pays a premium at time 0 and receives annual payments

time	0	1	...	n-1	n
cash flow	$-C_0$	c	...	c	c

- Equity: after purchase, you will receive a dividend at constant intervals

time	0	1	2	3	...
cash flow	$-C_0$	C_1	C_2	C_3	...

- Term assurance: premium paid at time 0. If death occurs before $t=n$, the beneficiary will receive an amount c.
- Loans and mortgages: borrower receives C_0 at $t=0$, then repays it (capital and interest) in a series of payments
- Insurance: company receives premium, with the amount and frequency of negative cash flows being uncertain.

Net present value and discounted cash flow

- The DCF formula states that the NPV of a series of cash flows is given by:

$$NPV(\underset{\sim}{c}) = c_0 + \frac{c_1}{1+i} + \frac{c_2}{(1+i)^2} + \dots \frac{c_n}{(1+i)^n} = \sum_{j=0}^{n} \frac{c_j}{(1+i)^j}$$

- Properties of NPV:
  - decreasing in i, and decreases faster when later payments are larger
  - if $a_j \geq b_j$, $j=0,\dots,n$, then $NPV(\underset{\sim}{a}) \geq NPV(\underset{\sim}{b})$ for $i \in [0,\infty)$

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- If $v(t)$ is the present value of a unit amount at time t,

$$NPV(\underset{\sim}{c}) = \sum_{j=1}^{n} c_j v(t_j)$$

- To find the NPV for continuous cash flows $\rho(t)$, we let $c_t = \rho(t)dt$ and take the limit of the above.

$$NPV(\rho) = \int_0^t \rho(u) v(u) du$$

- In the special case of constant interest i, we have:

$$NPV(\rho) = \int_0^t \frac{\rho(u)}{(1+i)^u} du$$

- For a general force of interest function:

$$NPV(\rho) = \int_0^t \rho(u) \exp\left(-\int_0^u \delta(s) ds\right) du$$

- The accumulated value is just $NPV \exp(\int_0^t \delta(s) ds)$, i.e:

$$A(t) = \int_0^t \rho(u) \exp\left(\int_u^t \delta(s) ds\right) du$$

The equation of value and IRR: comparing projects

- The equation of value / yield equation of i is generally given by $f(i)=0$, where $f(i) = NPV(\underset{\sim}{c}, \rho)$
- If $f(i)=0$ has exactly one root i_0, with $i_0 > -1$, then the internal rate of return (IRR), or the money weighted rate of return, is defined to be i_0. The IRR is the interest that zeroes the NPV.
- If $i >$ IRR, $NPV < 0$.
- Numerical methods for finding the IRR:
  - expand $1/(1+r)^j$ as $1 - jr$
  - multiply out $(1+r)^n$, then expand $(1+r)^j$ as $1 + jr + \frac{j(j-1)}{2} r^2$
  - assume all future cash-flows equal the average cash flow

t	0	1	2	...	$n-1$	n
C	$-C_0$	x	x	...	x	x

with $x = \sum_{j=1}^{n} c_j$

→

Then, we have $c_0 = \frac{x}{r}\left[1-\frac{1}{(1+r)^n}\right] = x\, a_{\overline{n}|,r}$, where $a_{\overline{n}|r} = \frac{1}{r}\left[1-\frac{1}{(1+r)^n}\right]$.
The values for $a_{\overline{n}|r}$ for different r can be found in tables.

- Distributing capital gains: if a cash flow consists of an interest payment x in years $0 \to n-1$ and a final payment of $c_n - x$, we can approximate $i = \dfrac{x + (c_n - x - c_0)/n}{c_0}$

· Disadvantages of IRR:
- there may be zero solutions or multiple solutions
- the IRR assumes that accumulated funds / loans will grow at the internal rate

· Projects ~~may~~ will have different profitability depending on interest rates. It may be useful to graph $NPV_i(\underset{\sim}{a})$ and $NPV_i(\underset{\sim}{b})$ as functions of i. The rate that gives equal NPVs is called the cross-over rate.

· The discounted payback period (DPP) is the smallest time such that the accumulated value of the project becomes positive
  - especially useful if capital is scarce
  - does not quantify profit: projects with late but large cash inflows may have higher DPP but also higher NPV.

· The payback period is the time when net cash flow is positive — it is inferior because it doesn't discount future cash flows

· Overall, NPV is the best metric when i is known. Sensitivity analyses should be performed to observe how NPV varies with i.

Measuring investment performance

- Many projects, e.g dividend-paying funds, will have cash flows of the form:

Time, t	t_0	t_1		t_n
Fund value at $t-0$	f_{0-0}	f_{1-0}	$\cdots$	f_{n-0}
Net cash flow at t	C_0	C_1		C_n
Fund value at $t+0$	$f_0 = f_{0-0} + C_0$	$f_1 = f_{1-0} + C_1$	$\cdots$	$f_n = f_{n-0} + C_n$

- The internal rate of return solves the equation of value, which equates all inflows with the eventual accumulation:

$$f_0 (1+i)^t + c_1 (1+i)^{t-1} + \dots + c_{n-1}(1+i) = f_n$$

- It is also called the money-weighted rate of return (MWRR) because it is influenced more by the performance when the fund is larger.
- The MWRR has some drawbacks:
  - sensitive to the amount and timing of cash flows, which are not controlled by fund managers
  - may not have a unique solution
- Alternatively, the time-weighted rate of return (TWRR) ignores the size of the fund and instead 'averages' the growth between cashflows

$$(1+i)^t = \frac{f_{1-0}}{f_0} \cdot \frac{f_{2-0}}{f_1} \cdots \frac{f_{n-0}}{f_{n-1}}$$

- However, the TWRR requires knowledge of all cash flows, and the value of the fund on the cashflow date.
- The Linked internal rate of return (LIRR) can be used to approximate the TWRR: it calculates the IRR at specified sub-intervals then combines them with the TWRR formula:

$$(1+i)^{t_n} = (1+i_1)^{t_1} (1+i_2)^{t_2-t_1} (1+i_3)^{t_3-t_2} \dots (1+i_n)^{t_n-t_{n-1}}$$

  - the LIRR can be used if the fund is not valued at every cashflow
  - LIRR = TWRR if the interval lengths are the same

3. Perpetuities, Annuities and Loans

Perpetuities

- A perpetuity is a special case of an annuity that pays a fixed amount every year forever.

time	0	1	2	3	...
cash flow	0	1	1	1	...

- The present value at time zero is denoted as $a_{\overline{\infty}|}$:

$$a_{\overline{\infty}|,i} = v + v^2 + v^3 + \dots + v^n = \frac{v}{1-v} = \frac{1}{i}$$

- The present value at time 1 is $\ddot{a}_{\overline{\infty}|}$. This quantity is equal to the value of the perpetuity with payments made in advance. Clearly $\ddot{a}_{\overline{\infty}|} = (1+i)\, a_{\overline{\infty}|}$

Annuities

- The most common form has constant annual payments

time	0	1	2	...	$n-1$	n
cash flow	0	1	1	...	1	1

- $a_{\overline{n}|}$ is the value at time zero, called the present value of the annuity-immediate a. This corresponds to payments in arrears.

$$a_{\overline{n}|} = v + v^2 + \dots + v^n = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{i}$$

- $\ddot{a}_{\overline{n}|}$ is the value at time 1, the present value of the annuity-due, which corresponds to payments in advance.

$$\ddot{a}_{\overline{n}|} = (1+i)\, a_{\overline{n}|} = \frac{1-v^n}{1-v}$$

- $s_{\overline{n}|}$ is the accumulated value of the annuity-immediate

$$s_{\overline{n}|} = (1+i)^n a_{\overline{n}|}$$

· A deferred annuity has its first payment at time $k+1$. Its present value is the same as that of a standard annuity, just scaled down by v^k, i.e: ${}_{k|}a_{\overline{n}|} = v^k a_{\overline{n}|}$

Annuities payable mthly

time	0	$1/m$	$2/m$	$\dots$	$(nm-1)/m$	nm/m	$(nm+1)/m$
C	0	$1/m$	$1/m$	$\dots$	$1/m$	$1/m$	$1/m$
	$a_{\overline{n}|}^{(m)}$	$\ddot{a}_{\overline{n}|}^{(m)}$				$s_{\overline{n}|}^{(m)}$	$\ddot{s}_{\overline{n}|}^{(m)}$

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\sum_{j=1}^{nm} v^{j/m} = \frac{1-v^n}{i^{(m)}} = \frac{i}{i^{(m)}}\, a_{\overline{n}|}$$

and $\ddot{a}_{\overline{n}|}^{(m)} = (1+i)^{1/m} a_{\overline{n}|}^{(m)}$

Increasing annuities

time	0	1	2	$\dots$	n
C	0	1	2	$\dots$	n
	$(Ia)_{\overline{n}|}$	$(I\ddot{a})_{\overline{n}|}$			

$$(Ia)_{\overline{n}|} = v + 2v^2 + \dots + nv^n = \frac{1}{i}\left[\frac{1-v^n}{1-v} - nv^n\right] = \frac{a_{\overline{n}|} - nv^{n+1}}{1-v}$$

General results about increasing annuities can be found by considering equivalent cash flows

	time	0	1	2	3	$\dots$	n
e.g	C	0	P	$P+Q$	$P+2Q$		$P+(n-1)Q$
$=$	C_1	0	$P-Q$	$P-Q$	$P-Q$	$\dots$	$P-Q$
	$+\ C_2$		Q	$2Q$	$3Q$	$\dots$	nQ

$$\therefore\ NPV = (P-Q)a_{\overline{n}|} + Q(Ia)_{\overline{n}|}$$

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Continuously payable annuities

$$\bar{a}_{\overline{n}|} = \int_0^n \rho(t) v(t)\,dt = \int_0^n v^t\,dt = \frac{1-v^n}{\delta} = \frac{i}{\delta} a_{\overline{n}|}$$

Many of the previous results apply:

- $\bar{s}_{\overline{n}|} = (1+i)^n \bar{a}_{\overline{n}|}$
- $_{k|}\bar{a}_{\overline{n}|} = v^k \bar{a}_{\overline{n}|}$
- $\bar{a}_{\overline{\infty}|} = \frac{i}{\delta} a_{\overline{\infty}|}$

For a continuously payable annuity increasing in steps:

$$(I\bar{a})_{\overline{n}|} = \sum_{j=1}^{n} \int_{j-1}^{j} j v^t\,dt = \frac{(1+i)\,a_{\overline{n}|} - n v^n}{\delta}$$

If payment increases continuously (i.e $\rho(t) = t$)

$$(\overline{Ia})_{\overline{n}|} = \int_0^n t v^t\,dt = \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta}$$

Loan schedules

Loan schedules are represented in tables:

year	loan outstanding before repayment	repayment	interest due	capital repaid	loan outstanding after repayment
1	l_0	x_1	$i l_0$	$x_1 - i l_0$	$l_1 = l_0 - (x_1 - i l_0)$
2	l_1	x_2	$i l_1$	$x_2 - i l_1$	$l_2 = l_1 - (x_2 - i l_1)$
...	...	...	...	...	...
k	l_{k-1}	x_k	$i l_{k-1}$	$x_k - i l_{k-1}$	$l_k = l_{k-1} - (x_k - i l_{k-1}$
...	...	...	...	...	...
n	l_{n-1}	x_n	$i l_{n-1}$	$x_n - i l_{n-1}$	$l_n = 0$

The equation of value for a loan is:

$$l_0 = x_1 v + x_2 v^2 + \ldots + x_n v^n$$

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There are two methods for finding l_k, the outstanding loan value immediately after the kth payment:

1. The retrospective loan calculation:

$$\underset{\text{after } k^{th} \text{ payment}}{\text{value of loan}} = \underset{\text{of original loan}}{\text{value at time } k} - \underset{k \text{ of repayments}}{\text{accumulated value at time}}$$

i.e $l_k = l_0(1+i)^k - \left[x_1(1+i)^{k-1} + x_2(1+i)^{k-2} + \ldots + x_k\right]$

2. The prospective loan calculation (generally easier):

$$\underset{\text{after } k^{th} \text{ payment}}{\text{value of loan}} = \underset{\text{of all future repayments}}{\text{value at time } k}$$

i.e $l_k = v x_{k+1} + v^2 x_{k+2} + \ldots + v^{n-k} x_n$

- In the special case of level payments (equal instalments), the equation of value is $l_0 = x a_{\overline{n}|}$
  - the prospective calculation shows that $l_k = x a_{\overline{n-k}|}$
  - the k^{th} payment consists of interest $x(1-v^{n-k+1})$ and a capital repayment of $x v^{n-k+1}$.
- For the more general case of mthly payments of $1/m \times x$, we have $l_0 = m x a^{(m)}_{\overline{n}|}$

Flat rate and APR

- Banks sometimes quote a flat rate of interest, which means that interest is charged on the full amount of the loan, regardless of capital repayments

  total interest paid = flat rate × size of loan × length of loan
- The effective rate tends to be just under double the flat rate.
- The annual percentage rate (APR) is the effective rate rounded to the lower 0.1%

4. Basic Financial Instruments

Markets, interest rates and instruments

- A primary market is where a security is originally issued.
- If the security is negotiable, it can subsequently be traded in a secondary market.
- The capital market refers to all long-term financial instruments
- A security is a bearer security if payment is made to whoever is holding it. Registered securities include a central record of ownership.
- The money market is the market in short-term (< 1 year) financial instruments which are based on an interest rate.
  - Treasury bill / T-bill: borrowing by the government, negotiable and bearer security. Carries a coupon. Called a Euro bill in Europe
  - Time deposit: a non-negotiable borrowing by a bank
  - Certificate of deposit (CD): negotiable bearer borrowing by a bank, usually carrying a coupon
  - Commercial paper (CP): negotiable bearer borrowing by company
  - Repurchase agreement (repo): non-negotiable security to borrow against a long term instrument.
  - Futures contract: a deal to buy or sell an instrument at a future date.
- Functions of the money market:
  - banks can lend to other banks with money shortages
  - companies may have surplus / shortage of money
- The Bank of England can lend money to eligible commercial banks by buying their financial instruments, at the Official Dealing Rate.
- Banks in the UK will lend to each other at the LIBOR.

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Fixed government borrowings

- A borrower who issues a bond agrees to pay interest at a specified rate (coupon payments) until the maturity/redemption date, at which time a fixed redemption value is paid.
- The coupon rate is applied to the face/par value of the bond, which is the value that should be paid at maturity.
  - if face value = redemption value, the bond is redeemable at par
  - if face value < redemption, the bond is redeemable at a premium
  - if face value > redemption, the bond is redeemable at a discount
- Some bonds have variable redemption dates - the borrower decides.
- Some government bonds have coupon and redemption linked to an inflation index.
- Investment banks can decompose a government bond into strips (separately traded and registered interest and principal security).
- A UK government bond is called a gilt.

Corporate / bank fixed borrowings

- Debentures (corporate bonds) are issued by companies and are usually secured against specific assets. Because they are more risky, investors will expect a higher yield.
- Foreign bonds are issued in the local currency but by a foreign borrower, e.g:
  - Samurai bonds in Japan (yen)
  - Yankee bonds in the US
  - Bulldog bonds in the UK (£)
- Eurobonds are issued in a different currency in a different country e.g if I need USD in the Swiss financial market.
- Eurobonds were developed to make eurocurrency (currency deposited outside of its home country) lending more marketable.

- Certificates of deposit (CDs) are issued by banks in return for a deposit. There is an active secondary market, but this flexibility reduces yield.
  - because they are short term, simple interest is typically used
  - the maturity proceeds are calculated by simple accumulation of the face value with coupon rate i_c over d days

$$m_p = f\left(1 + \frac{i_c d}{d_T}\right)$$

d_T is 360 for ACT/360, 365 for ACT/365

  - if the CD is bought at time d_a for price p_a, and subsequently sold at time d_b for price p_b, the yield is

$$\left(\frac{p_b}{p_a} - 1\right)\frac{d_T}{d_b - d_a}$$

can be rearranged in terms of the yields at time d_a and d_b.

Investments with uncertain returns

- Ordinary shares entitle holders to a share in the net profits of a company, the dividend.
  - they are the last to be repaid during bankruptcy, so are the most risky
  - if a share is bought ex-dividend, the seller receives the next dividend payment.
- Preference shares normally pay a fixed dividend, and have a higher debt priority than ordinary shares
- Convertibles are loans that can be converted into ordinary shares at a fixed price and a fixed date
  - investors accept a lower interest payment because of possible capital gains from the rise in share prices
  - companies don't have to immediately dilute earnings and dividends as would be the case for issuing shares.
- Properties provide uncertain rental incomes and volatile capital gains.
- The running yield of an investment is the annual income divided by the current market price (ignoring capital gains). Thus:

bonds > property > equity.

Derivatives

- Derivatives are instruments whose value depends on other financial assets. They can be used to hedge positions or for speculation
- Market risk is the risk that market conditions change adversely, while credit risk is the risk that the counterparty in an agreement will default on its payments.

- A forward contract is a legally binding contract to buy/sell an agreed quantity of an asset at a specified time in the future.
- A futures contract has the same definition; the difference is that forward contracts are over-the-counter (OTC), while futures are traded on an exchange and are thus more standardised.
- At maturity, the short side (seller) delivers the asset to the long side (~~byer~~ buyer) for the exchange delivery settlement price (EDSP)
- Often, a cash settlement is made instead of actually transfering assets.
- Each party in a futures contract must deposit some money, called the margin, to the clearing house to control credit risk.

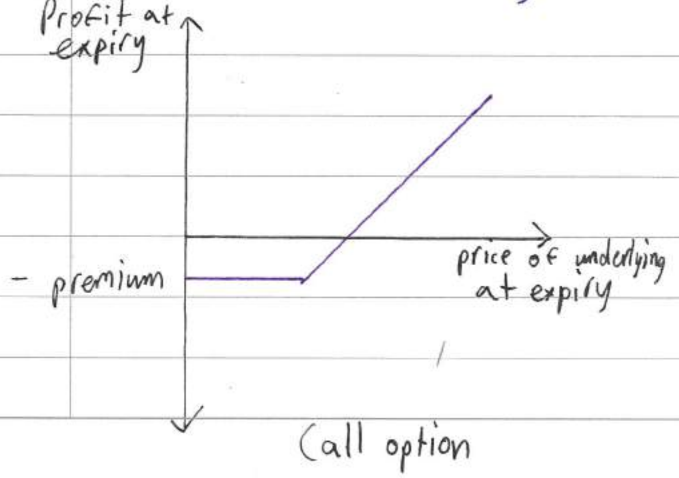
- A financial future is a contract based on a financial asset.
- A bond future involves a notional bond - the exact list of eligible bonds must be agreed on. At maturity, the short side will deliver the cheapest bond.
- Stock index futures can be used to speculate on the movement of the market as a whole.
- Currency futures require the delivery of a specified amount of a given currency on a specified date.
- A forward rate agreement (FRA) is an interest rate hedge in which the buyer (normally a borrower) fixes the interest rate on his loan, while the seller receives the actual floating rate. Thus a buyer of an FRA profits if interest rates rise.

- An interest rate future is a futures-contract equivalent of an FRA. The price of an interest rate future rises as interest rates fall; they are usually priced at $100 - i$.

- A currency swap is an agreement to exchange interest payments and a capital sum of one currency for those in another currency. This is useful because some parties may be able to get favourable rates in their home country.
- In an interest rate swap, two parties swap a series of interest payments. Usually, one party will want a fixed instead of floating rate.
  - with a cap, the bank agrees to pay any interest above a certain rate
  - with a floor, the bank agrees to always pay you at least a certain amount of interest
  - a combination of cap and floor is called a collar.

Options

- A call option gives the holder the right to to buy an asset for a specified price (the strike price) at a specified time in the future
- A put option lets you sell at the strike price.
- An American option can be exercised at any date before expiry, while a European option can only be exercised on the expiry date.
- An option is in-the-money if immediate execution would be profitable, and out-of-the-money otherwise.



Call option

- The maximum loss for the holder of an option is the premium paid.
- For a call, the holder breaks even if the price of the underlying at expiry equals the exercise price plus the premium paid.

5. Bonds, equities and inflation

Bonds

- A borrower who issues a bond pays a coupon on the face value of the bond at specified intervals until maturity, at which time they pay a fixed redemption value (almost always the same as the face value).

Fixed by the terms:
f = face/par value
r = coupon rate per year
C = redemption value

Vary throughout bond's lifetime:
P = current price
i = yield to maturity. (same as IRR).
n = time to redemption

- By writing down the cashflows of the bond, it can be seen that:

$$P = fr\, a^{(m)}_{\overline{n}|,i} + Cv^n$$

  - this is the equation of value
  - price varies inversely as yield, given fixed n.
  - if $P = f = C$, then $i^{(m)} = r$.

- If income (coupon payments) are taxed at rate t_1:

$$P = fr(1-t_1)\, a^{(m)}_{\overline{n}|,i} + Cv^n$$

- Ratio of coupon to current price has many names: gross interest yield, flat yield, direct yield, current yield, gross running yield. These exclude tax. Net interest yield or net running yield includes tax.
- The yield to maturity, also called the gross/net yield to redemption, includes all subsequent coupons as well as the final redemption.
- To calculate yield to maturity, we can approximate it as being equal to (coupon per year + capital gains per year)/price, i.e:

$$i \simeq \frac{1}{P}\left(fr + \frac{100-P}{n}\right)$$ overestimates i.

- For simple cases, if $i > r$ then the bond is trading at a discount, and if $i < r$ then the bond is trading at a premium.
- To generalise slightly, let $g = fr/c$ (reduces back to the coupon rate if bond is redeemed at par). Then:

  if $i^{(m)} \leq (1-t_1)g$, it means that $P \geq C$ (premium)

  $i^{(m)} \geq (1-t_1)g \Rightarrow P < C$ (discount).
- In the latter case, capital gains tax (t_2) applies:

$$P = \frac{(1-t_1) fr a_{\overline{n}|}^{(m)} + (1-t_2) C v^n}{1 - t_2 v^n}$$

- Some bonds have their redemption date variable (at the bond issuer's option). The investor cannot know the exact yield, and must assume the worst case:
  - if $i^{(m)} < (1-t_1)g$, there is no capital gain and should thus expect earliest possible redemption
  - if $i^{(m)} > (1-t_1)g$, assume latest redemption as issuer would like to defer paying out.

- If a bond is sold between coupon dates, the seller feels entitled to some accrued interest. He thus sells the bond for the dirty price:

  dirty price = NPV of future cash flows

  clean price = dirty price − accrued interest
- Under ACT/365:

$$\text{accrued interest} = \text{annual coupon} \times \frac{\text{days since last coupon}}{365}$$

- Bonds quoted cum dividend mean that the next coupon will go to the buyer
- Ex dividend means that the buyer will not receive the coupon, so accrued interest is negative.

Date NO.

Equity calculations and interest with inflation

- Equities are characterised by a stream (often assumed to be infinite) of uncertain dividend payments.
- The money rate of interest of an investment ignores ~~interest~~ inflation; the effective rate of interest includes the inflation adjustment.

$$1 + i_R = \frac{1+i_M}{1+q} \quad , \text{ where } q \text{ is the inflation rate.}$$

- For a general cash flow:

Time	0	1	2	...	n
cash flow	$-P$	C_1	C_2	...	C_n
inflation	$Q(0)$	$Q(1)$	$Q(2)$	...	$Q(n)$

The equation of value is given by:

$$P = \sum_{k=1}^{n} C_k \frac{Q(0)}{Q(k)} v^k \quad , \text{ with } v = \frac{1}{1+i}$$

- For the special case of constant inflation

$$P = \sum_{k=1}^{n} C_k v_M^k = \sum_{k=1}^{n} \frac{C_k}{(1+q)^k} v_R^k \quad , \text{ using } v_R = (1+q) v_M$$

- Some securities have their payments / coupons linked to an inflation index. In theory, this means that $C_k' = C_k \, Q(k)/Q(0)$, but in practice there may be a lag time.

6. <u>Interest Rate Problems</u>

Spot rates, forward rates and the yield curve

- The term structure of interest rates is the relationship between interest rates and maturities.
- Instead of a growth $(1+i)^t$, we have i as a function of t, so the accumulation will be $(1+y_t)^t$.
- y_t is called the t-year spot rate of interest
- Any fixed interest security can be analysed in terms of spot rates by looking at it as a sum of zero-coupon bonds.
- For example, a bond with coupon x annually and redemption C obeys:

$$P = x\left[\frac{1}{(1+y_1)} + \frac{1}{(1+y_2)^2} + \dots + \frac{1}{(1+y_n)^n}\right] + \frac{C}{(1+y_n)^n}$$

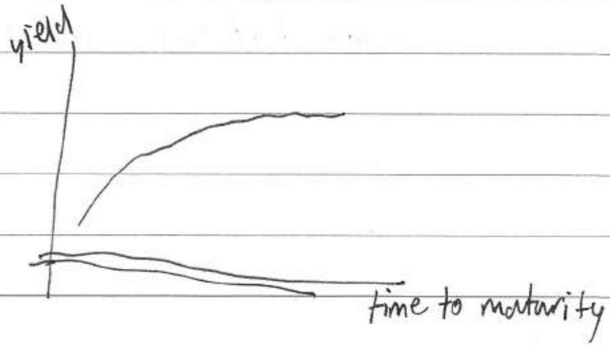
- y_1 can be determined from the yield of 1y T Bills, then y_2 from the price of 2y bonds, and so on:

$$P = \frac{fr}{(1+y_1)} + \frac{fr+C}{(1+y_2)^2} \leftarrow y_1 \text{ known, so } y_2 \text{ can be found.}$$

- A zero coupon bond can also be constructed by subtracting bonds with different coupons.
- The n-year par yield of a bond is the coupon rate that causes the bond price to be equal to its face value, assuming redemption at par. If r is the par yield, r satisfies:

$$r\sum_{k=1}^{n}\frac{1}{(1+y_k)^k} + \frac{1}{(1+y_n)^n} = 1$$

- The coupon bias is the difference between the par yield and the spot rate.
- Plotting y_t against t gives the yield curve.



- The forward interest rate $f_{T,k}$ is the interest on money borrowed from time T to time $T+k$.
  - like the spot rate, but doesn't necessarily start at $t=0$.
  - $f_{0,k} = y_k$
  - sometimes $f_{T,k}$ means money borrowed from T to k.
  - the forward rate ~~is the ge~~ can be calculated from spot rates

$$(1+y_2)^2 = (1+y_1)(1+f_{1,1})$$
$$(1+y_3)^3 = (1+y_2)^2(1+f_{2,1}) = (1+y_1)(1+f_{1,1})(1+f_{2,1})$$
$$\dots$$
$$(1+y_k)^k = (1+f_{0,1})(1+f_{1,1})\dots(1+f_{k-1,1})$$

  - i.e the spot rate is the geometric mean of 1y forward rates.

- In the continuous case, we have the spot force of interest, and the forward force of interest (denoted by Y_t and F_t)
  - $(1+y_t)^t = e^{tY_t}$
  - $(1+f_{t,k})^k = e^{kF_{t,k}}$
- The instantaneous forward rate can be found by taking the limit $\lim_{k\to 0} F_{T,k}$, leading to $P_t = \exp\left(-\int_0^t F_t \, dt\right)$

- Expectations theory explains the yield curve in terms of expected future movements in interest rates: interest rate expected to increase $\Rightarrow$ long term bonds less attractive $\Rightarrow$ yields increase to compensate
- Liquidity Preference theory: short term bonds are more flexible and inherently less risky as a result. So long term bonds must offer higher yield.
- Market segmentation theory suggests that there are different forces of supply and demand for bonds of different lengths: e.g banks want short term, institutions want long term, etc.

Vulnerability to interest rate movements

- If P is the present value of a series of cashflows, then the Macaulay duration / duration / discounted mean term is given by:

$$d_M(i) = -\frac{1}{P}\frac{dP}{d\delta} = \frac{1}{P}\sum_{k=1}^{n}\frac{t_k c_{t_k}}{(1+i)^{t_k}}$$

  - the duration is the mean term of the cash flows weighted by their PVs. It has units of years
  - the duration measures sensitivity to interest rates
  - as the yield decreases, duration increases.

- The effective duration / modified duration / volatility of a cash flow is defined as: $d(i) = -\frac{1}{P}\frac{dP}{di} = \frac{1}{P}\sum_{k=1}^{n}\frac{t_k c_{t_k}}{(1+i)^{t_k+1}}$
  - it is related to the Macaulay duration by $d_M(i) = (1+i)d(i)$
  - if the yield decreases by ~~a factor of~~ ε, price increases by a factor of $\varepsilon d(i)$.
- The convexity of a cash flow is the (positive) second derivative of price w.r.t yield, per unit price.

$$c(i) = \frac{1}{P}\frac{d^2P}{di^2} = \frac{1}{P}\sum_{k=1}^{n}\frac{t_k(t_k+1)c_{t_k}}{(1+i)^{t_k+2}}$$

  - higher positive convexity is good for the investor
  - convexity is related to variance: a more spread out cash flow is usually more convex.

- A fund has Redington immunisation at i_0 subject to 3 conditions:

1. NPV(assets) = NPV(liabilities)
2. d_M(assets) = d_M(liabilities)
3. c(assets) > c(liabilities)

- Condition 3 is equivalent to: $\sum_{k=1}^{n} t_k^2 a_{t_k} v^{t_k} > \sum_{k=1}^{n} t_k^2 l_{t_k} v^{t_k}$

Stochastic interest rate problems

- If i_t is the interest rate from time $t-1$ to t, the accumulated value of a unit investment at time n is:

$$S_n = (1+i_1)(1+i_2)\ldots(1+i_n)$$

- If i_t is a random variable with mean μ and variance σ^2, and $i_1, i_2, \ldots i_n$ are independent:

$$E(S_n) = (1+\mu)^n$$
$$E(S_n^2) = \left[(1+\mu)^2 + \sigma^2\right]^n$$
$$\therefore Var(S_n) = \left[(1+\mu)^2+\sigma^2\right]^n - (1+\mu)^{2n}$$

- If there are multiple investments, $E(A_n)$ and $E(A_n^2)$ can be calculated by noting that $A_n = (1+i_n)(1+A_{n-1})$.

- For a random variable Y, if $Y \sim N(\mu, \sigma^2)$, then:

$$\ln Y \sim \text{Lognormal}(\mu, \sigma^2)$$ not the mean/var of the lognormal dist!

- If $Y_i \sim \text{lognormal}(\mu, \sigma^2)$, $\prod_{i=1}^{n} Y_i \sim \text{lognormal}(n\mu, n\sigma^2)$, i.e the product of lognormal distributions is a lognormal variable.
- Expectation of a lognormal dist $= e^{\mu + \frac{1}{2}\sigma^2}$
- Variance $= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$
- Probability calculations will require the table of $\phi(x)$, usually with interpolation.

7. Arbitrage

- If investment A costs S_A and B costs S_B, arbitrage exists in any of the following cases:
  - $NPV(A) = NPV(B)$ and $S_A \neq S_B$
  - $NPV(A) \neq NPV(B)$ and $S_A = S_B$
  - $S_A > S_B$ and $NPV(A) < NPV(B)$
- Realising the arbitrage involves buying some of A and shorting some B.

- A forward contract specifies that A will buy asset S from B for price K, at time T.
- In the simplest case of a constant force of interest with no income, the contract can be priced by considering the NPV

$$S_0 = Ke^{-\delta T}$$

- If there are fixed coupons, we can amend the above to include $ce^{-\delta t_i}$.
- However, it is different when the dividend yield is paid on the (unknown) price of the asset. By the no-arbitrage assumption, at time T we require that $S = K$. Then we have:

$$S_0 e^{-rT} = Ke^{-\delta T}$$

- The value of a long contract at an intermediate time can be calculated by finding $K_t - K_0$, where t is the time of valuation, then discounting.

$$V_t = S_t - S_0 e^{\delta t}$$