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Analysis

Series

- A sequence f_n converges to the limit L as $n \to \infty$ if $|f_n - L| < \varepsilon$ for sufficiently large n.
- Cauchy's principle of convergence is that $|f_{n+m} - f_n| < \varepsilon$ for all $m \in \mathbb{Z}$ if n is sufficiently large (necessary and sufficient)
- The convergence of an infinite series depends on its partial sum:
  - ↳ if $\sum |u_n|$ converges, the series is absolutely convergent
  - ↳ if $\sum |u_n|$ diverges but $\sum u_n$ converges, the series is conditionally convergent
- Necessary condition for convergence: $u_n \to 0$ as $n \to \infty$
- Comparison test: if $|v_n|$ converges and $|u_n| \leq k|v_n|$, then $|u_n|$ converges. (likewise for divergence).
- Ratio test: let $r = \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right|$
  - ↳ if $r < 1$, $\sum u_n$ converges absolutely
  - ↳ if $r > 1$, $\sum u_n$ diverges
  - ↳ if $r = 1$, inconclusive
- Cauchy's root test: similar to ratio test except $r = \lim_{n\to\infty} |u_n|^{1/n}$

Complex analysis

- The derivative of $f(z)$ at $z = z_0$ is:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

  ↳ this limit must be the same when approaching z_0 from any direction in the complex plane
- Consider $f(z) = u(x,y) + iv(x,y)$. If $f'(z)$ exists, we should be able to approach it along either the real or imaginary axes

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

from definition of derivative.

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

  ↳ these are the Cauchy-Riemann equations
  ↳ they are necessary and sufficient conditions for $f'(z)$ to exist (if partials are continuous).
- A function is analytic in a region R if $f'(z)$ is defined for $\forall z \in R$. If R is the entire complex plane, f is entire.
  - ↳ sums, products, and compositions of analytic functions are also analytic
  - ↳ a function is analytic at a point if $f(z)$ is differentiable in a small neighbourhood around z_0.

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- Many complex functions are analytic everywhere except at certain points - singularities. e.g $f(z) = P(z)/Q(z)$ has singularities at $Q(z) = 0$
- If we know a function is analytic, the CR equations can tell us the imaginary part (within a constant) if we knew the real part.
- The real and im parts both satisfy Laplace's equation:
$$\nabla^2 u = \nabla^2 v = 0$$
- The curves of constant u are orthogonal to the curves of constant v: $\nabla u \cdot \nabla v = 0$

Power Series

- If a function is analytic in R, it is infinitely differentiable everywhere in R. Thus it can be expressed as an infinite Taylor series - this is an alternate definition for analyticity:
$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$
- A zero of $f(z)$ is of order N if
$$f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{(N-1)}(z_0) = 0$$
but $f^{(N)}(z_0) \neq 0$
  - ↳ i.e the first nonzero term in the Taylor series is proportional to $(z - z_0)^N$.
- A pole is like a vertical asymptote. If $g(z)$ is analytic and nonzero at $z = z_0$, then $f(z)$ has a pole of order N where: $f(z) = \dfrac{g(z)}{(z - z_0)^N}$
  - ↳ if $f(z_0)$ is a zero of order N, then $1/f(z_0)$ is a pole of order N
  - ↳ i.e the number of times you must multiply a function by $(z - z_0)$ to make it analytic.
  - ↳ if $N \to \infty$, $f(z)$ has an essential singularity

- Behaviour for $z \to \infty$ is examined by considering $g(\xi) = f\left(\frac{1}{\xi}\right)$ then analysing $\xi \to 0$.

Laurent Series

- Any function that is analytic and single-valued through an annulus $a < |z - z_0| < b$ centred on $z = z_0$ has a unique Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \Big\} \text{ more general than Taylor}$$

↳ if the first nonzero term has $n \geq 0$, this is just a Taylor series about z_0 so f is analytic at $z = z_0$

↳ if the first nonzero term is for some $n = -N < 0$, $f(z)$ has a pole of order N at z_0.

↳ if there are an infinite number of terms, $f(z)$ has an essential singularity.

e.g. $e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$, so there is an essential singularity at $z = 0$

Convergence of power series

- If a power series $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges for $z = z_1$, it must converge absolutely for all $|z - z_0| < |z_1 - z_0|$

- Hence there exists a radius of convergence R such that the series: ⇐ may be zero or infinite

↳ converges for $|z - z_0| < R$

↳ diverges for $|z - z_0| > R$

↳ may converge or diverge on the circle of convergence $|z - z_0| = R$

- The ratio of terms in the power series is $r_n = \left|\frac{a_{n+1}}{a_n}\right| |z - z_0|$

↳ by the ratio test, if $|a_{n+1}/a_n| \to L$ as $n \to \infty$, the series converges for $L|z - z_0| < 1$, so the radius of convergence is $1/L$

- Alternatively, the radius of convergence is equal to the nearest singular point ← where not analytic

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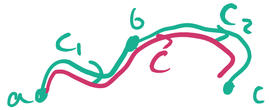
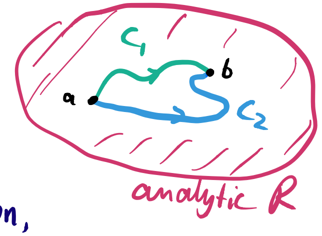
Contour Integration

- The integral along a contour C in the complex plane is defined as:
$$\int_C f(z)dz \equiv \lim_{|\delta z| \to 0} \sum_{k=0}^{N-1} f(z_k)\delta z_k \leftarrow \text{an element of the contour}$$
- The result may depend on the contour, and direction matters
- Contours can be added and subtracted
$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz.$$



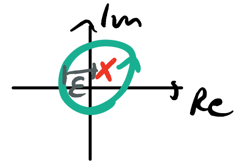
- For a closed contour $\oint_C f(z)dz$, it doesn't matter where we start but direction matters.
- A simple closed curve is continuous, has finite length, and does not intersect itself. It partitions the complex plane into interior/exterior.
- Cauchy's theorem states that if $f(z)$ is analytic in a simply-connected domain R, then for any simple closed curve C in R, $\oint_C f(z)dz = 0$.
  - ↳ the proof requires Green's theorem (2D Stokes), i.e
  $$\oint_C (w_x dy - w_y dx) = \int_S \left(\frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y}\right) dx dy.$$
  - ↳ expand f and dz then apply the C-R equations
  $$\begin{aligned}\oint_C f(z)dz &= \oint_C (u+iv)(dx+idy) \\ &= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \\ &= -\int_S \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dxdy + i\int_S \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \\ &= 0\end{aligned}$$
  Green's thm
  C-R.

- Cauchy's theorem implies that we can deform a contour without changing the value of $\int_C f(z)dz$, provided that we do not cross a singularity
  - ↳ for contours C_1, C_2 from $a \to b$, $C \equiv C_1 - C_2$ is closed.
  - ↳ hence as long as f is analytic in the region, $\oint_C f(z)dz = 0 \Rightarrow \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$.
  - ↳ For an entire function, contour integration is path-independent.



Residues

- Given the Laurent series of a function $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$, the residue of a pole is the coefficient a_{-1}
  - ↳ if there is a simple pole at z_0: $\operatorname{res}_{z=z_0} f(z) = a_{-1} = \lim_{z \to z_0} \{(z-z_0)f(z)\}$
  - ↳ for a pole of order N:
  $$\operatorname{res}_{z=z_0} f(z) = a_{-1} = \lim_{z \to z_0} \left\{ \frac{1}{(N-1)!} \frac{d^{N-1}}{dz^{N-1}} \left[(z-z_0)^N f(z)\right] \right\}$$
  - ↳ L'Hôpital's rule is often used to compute residues.
  - ↳ if $f(z)$ has a simple zero at $z=z_0$, $\operatorname{res}_{z=z_0} \frac{1}{f(z)} = \frac{1}{f'(z_0)}$
- Consider the contour integral around a pole:
  - ↳ $\oint_C f(z)dz = \oint_C \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n dz$
  - ↳ for $n \geq 0$, $\oint_C a_n (z-z_0)^n dz = 0$ (analytic)
  - ↳ for $n < 0$ we shrink the contour to a circle of radius ε and use $z = z_0 + \varepsilon e^{i\theta}$
  $$\Rightarrow \oint_{C'} a_n (z-z_0)^n dz = \begin{cases} 2\pi i a_{-1}, & n = -1 \\ 0, & n \neq -1 \end{cases}$$

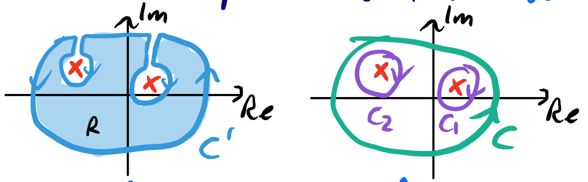


↳ reordering the sum and integral, $\oint_C f(z)dz = 2\pi i \operatorname{res}_{z_0} f(z)$

- The residue theorem states that if $f(z)$ is analytic in a simply-connected R except for a finite number of poles at $z = z_1, \dots z_n$, and C is a simple closed curve that encircles the poles in a positive sense (anticlockwise):

$$\boxed{\frac{1}{2\pi i}\oint_C f(z)dz = \sum_{k=1}^{n} \operatorname{res}_{z_k} f(z)}$$

↳ this follows from the previous result for $\oint_C f(z)dz$ with a pole.



↳ $\oint_{C'} f(z)dz = \oint_C f(z)dz + \sum_k \oint_{C_k} f(z)dz$ ← joining lines cancel

↳ but $\oint_{C'} f(z)dz = 0$ by Cauchy's theorem, since R does not contain any poles.

↳ $\therefore \oint_C f(z)dz - 2\pi i \sum_k \operatorname{res}_{z_k} f(z) = 0$, from which we get the residue theorem.

- If $f(z)$ is analytic in R containing z_0, $\frac{f(z)}{z-z_0}$ is analytic except for a simple pole at $z = z_0$ with residue $f(z_0)$. Applying the residue theorem gives Cauchy's formula

$$\boxed{f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz}$$

↳ if we know $f(z)$ on C, we know $f(z)$ in the interior too

↳ this is equivalent to the uniqueness theorem.

- Be careful when applying the residue theorem to points at infinity. Using $\zeta = \frac{1}{z}$, $\frac{dz}{z} = -\frac{d\zeta}{\zeta}$

Computing integrals using residues

- For trig functions, sub $z = e^{i\theta}$ and write trig functions in terms of z, e.g. $dz = iz\,d\theta$, $\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

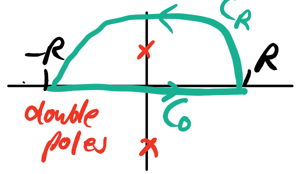
  ↳ we may then be able to identify poles

  ↳ use the residue theorem (only considering poles inside C) to compute the integral

- For integrals with infinite bound, we will need to expand the contour to infinity.

  ↳ e.g. $I = \int_0^\infty \frac{1}{(x^2+a^2)^2}\,dx$

  ↳ consider a semi-circular contour



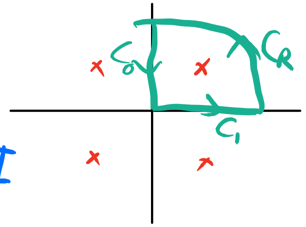
  ↳ by symmetry, $\int_{C_0} \frac{dz}{(z^2+a^2)^2} = 2\int_0^R \frac{dz}{(z^2+a^2)^2} = 2I$ as $R \to \infty$

  ↳ for the curved portion, the integrand is $O(R^{-4})$ while the contour has length πR, so $\oint_{C_R} f(z)dz \to 0$ as $R \to \infty$.

  ↳ computing the residue: $2I = 2\pi i\left(\frac{-i}{4a^3}\right) \Rightarrow I = \frac{\pi}{4a^3}$ //

- We can use different contours (though circular sectors are easier). e.g. for $I = \int_0^\infty \frac{1}{1+x^4}\,dx$

  ↳ $\oint_{C_R} f(z)dz \to 0$ as $R \to \infty$



$\Rightarrow \oint_C \frac{1}{1+z^4}\,dz = \int_0^R \frac{dx}{1+x^4} + \int_R^0 \frac{i\,dy}{1+(iy)^4} = (1-i)I$

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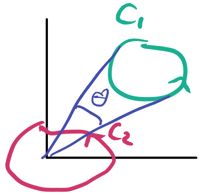
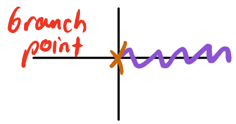
Multi-valued functions

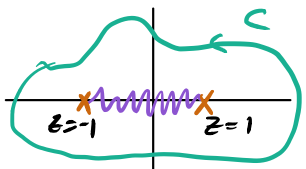
- Some functions e.g $\ln z$ are multi-valued for certain contours.
  - ↳ $\ln z$ has a branch point at the origin; we $+2\pi$ every time we circle it



  - ↳ but for a curve C_1, θ is in a definite range so $\ln z$ is continuous and single valued.
- We can introduce a branch cut to prevent curves from crossing a point

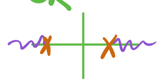


  - ↳ infinitely many possible cuts - conventional to choose axis when possible.
  - ↳ a branch of the function is then given by the domain $0 \le \theta < 2\pi$ around the branch point.
- Branch cuts prevent us from using Laurent series since the function cannot be analytic in an annulus
- $f(z) = (z-c)^{\alpha}$ has a branch point at c and, if α is rational, a finite number of branches.
- e.g $f(z) = \sqrt{z^2-1} = \sqrt{z-1}\sqrt{z+1}$ has branch points $z = \pm 1$
  - ↳ let $z-1 = r_1 e^{i\theta_1}$; $z+1 = r_2 e^{i\theta_2}$
    $\Rightarrow f(z) = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$

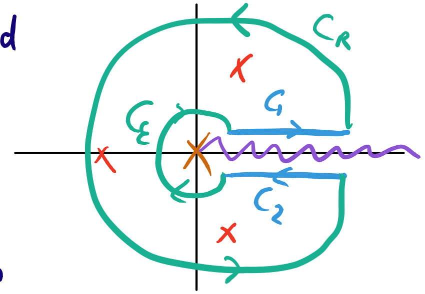


  - ↳ if C_1 encircles $z=1$, $\theta_1 \to \theta_1 + 2\pi$
  - ↳ if C_2 encircles $z=-1$, $\theta_2 \to \theta_2 + 2\pi$
  - ↳ if C encircles both or neither, no change

OR



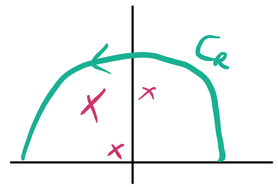
- To evaluate contour integrals around branch cuts, we consider keyhole contours in the limits $\varepsilon \to 0$, $R \to \infty$



  - ↳ in these limits, $C_\varepsilon \to 0$, $C_R \to 0$
  - ↳ we are left with C_1, C_2 which do not cancel because of the branch cut.
  - ↳ for C_1, $z = re^{i\theta}$ while for C_2, $z = re^{i\theta + 2\pi i}$
  - ↳ we can then apply the residue theorem as before.

Jordan's lemma

- Consider $I = \lim_{R\to\infty} \int_{C_R} f(z) e^{i\lambda z} dz$



  - ↳ $\lambda \in \mathbb{R}$, $\lambda > 0$
  - ↳ $f(z)$ analytic except for finite no. of poles
  - ↳ C_R is a semicircle in the upper half-plane
- Jordan's lemma states that if $\max |f(z)| \to 0$ as $R \to \infty$:

$$\boxed{\lim_{R\to\infty} \int_{C_R} f(z) e^{i\lambda z} dz = 0}$$

  - ↳ if $\lambda < 0$, we use a semicircle in the lower half-plane.
  - ↳ proof - let $z = Re^{i\theta}$ and $M \equiv \max_{C_R} |f(z)|$

$$\left| \int_{C_R} f(z) e^{i\lambda z} dz \right| \le M \int_0^{\pi} |e^{i\lambda z}|\,|Re^{i\theta}|\, d\theta$$

$$= M \int_0^{\pi} R e^{-\lambda y}\, d\theta$$

symmetry of $\sin\theta$ about $x = \pi/2$ ↳

$$= 2MR \int_0^{\pi/2} e^{-\lambda y}\, d\theta$$

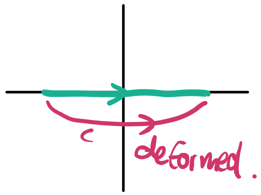
$y = R\sin\theta$ ↳

$$= 2MR \int_0^{\pi/2} e^{-\lambda R \sin\theta}\, d\theta$$

↳ $\sin\theta$ is concave on $[0, \frac{\pi}{2}]$ $\Rightarrow$ $\frac{2}{\pi}\theta \leq \sin\theta \leq 1$

$$\therefore \left| \int_{C_R} f(z) e^{i\lambda z} dz \right| \leq 2MR \int_0^{\pi/2} e^{-2\lambda R\theta/\pi} d\theta$$
$$= \frac{\pi}{\lambda}(1 - e^{-\lambda R}) M$$
$$\to 0 \text{ as } R \to \infty. \quad \text{QED}$$

e.g. Evaluate $I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$.

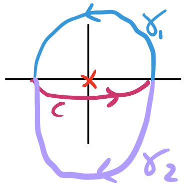


- Well-behaved at origin, so can integrate along real axis
- $I = \frac{1}{2i}\left[\int_C \frac{e^{iz}}{z} dz - \int_C \frac{e^{-iz}}{z} dz \right]$

$$\equiv \frac{1}{2i}[I_1 + I_2].$$ To evaluate these, deform the contour.

- There is now a pole at the origin. Add a large outer semicircle so we have a closed contour.
  - ↳ $\int_{C+\gamma_1} \frac{e^{iz}}{z} dz = 2\pi i$
  - ↳ $\int_{C+\gamma_2} \frac{e^{-iz}}{z} dz = 0$

  } by the Residue thm.

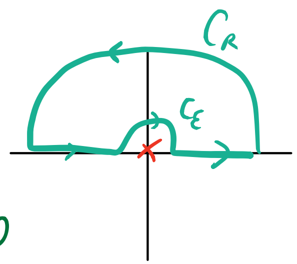


- But using Jordan's lemma, the integral along $\gamma_1, \gamma_2 \to 0$ as $R \to \infty$

$$\Rightarrow I = \frac{1}{2i}[2\pi i + 0] = \pi$$

- This integral can also be solved by noting

$$I = \text{Im}\left[\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz \right]$$



↳ Cauchy's theorem gives

$$\left[\int_{-R}^{-\varepsilon} dz + \underbrace{\int_{C_\varepsilon} dz}_{-i\pi} + \int_{\varepsilon}^{R} dz + \underbrace{\cancel{\int_{C_R} dz}}_{\text{Jordan's lemma}} \right] \frac{e^{iz}}{z} = 0$$