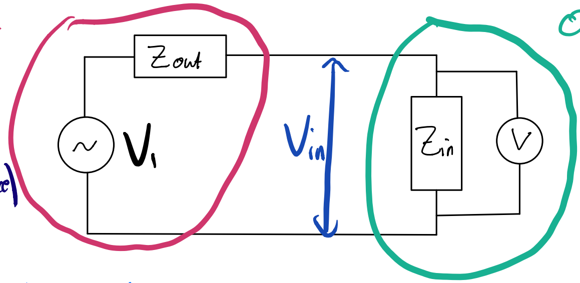
Robert Andrew Martin

Experimental Methods

System $\xrightarrow[\text{effect}]{}$ Transducer $\xrightarrow[\text{electrical signal}]{}$ Signal handling → Data

- The transducer mustn't affect the system or the effect.
- Oscilloscopes can measure time-varying voltages well:

Transducer
- V_1 is modeled as a perfect source (no impedance)



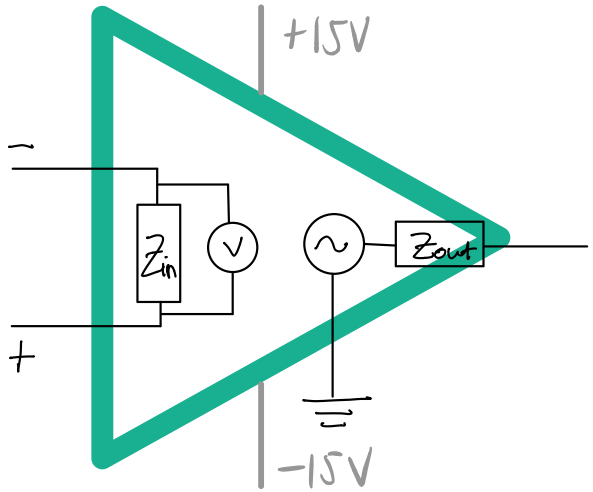
Oscilloscope
- Real scope modeled as an impedance with an ideal Ⓥ

- Ideally, $V_{in} = V_1$, i.e scope exactly measures transducer.
  ↳ but applying the potential divider equation gives:

$$V_{in} = V_1 \frac{Z_{in}}{Z_{in} + Z_{out}}$$

  ↳ i.e need scope with large impedance
  ↳ and transducer with very low impedance (large current).
- To compensate for the complex part of Z_{in}, we can add a capacitor to Z_{out}, in the scope probe.
- Current measurement requires low Z_{in}.

Amplifiers



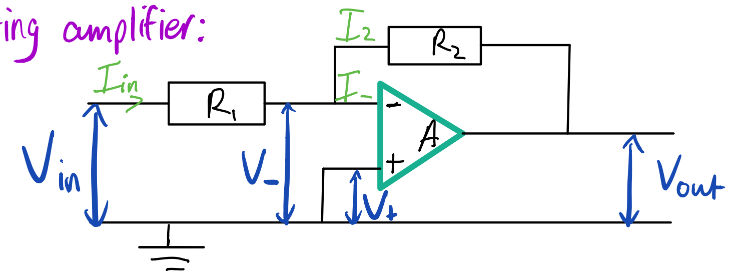
- The amplifier has an open loop gain of A:

$$V_{out} = A(V_+ - V_-)$$ → i.e amplify the difference

- For an ideal op-amp, we assume:
  - $A = \infty$
  - $Z_{in} = \infty$, i.e input current is irrelevant
  - $Z_{out} = 0$, i.e it can provide large current
- $A = \infty$ is ok because of negative feedback
  ↳ pass some output back to the −ve pin
- Golden rules:

1. Inputs draw no current (Z_{in} infinite)
2. Voltages on + and − pins are equal, if we provide negative feedback

- Inverting amplifier:



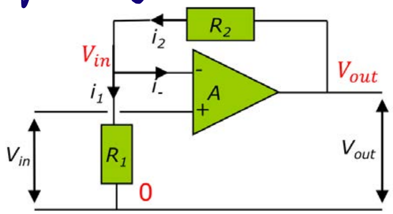
- $V_+ = 0$ because the circuit is grounded. $\Rightarrow V_- = 0$
- $I_- = 0 \Rightarrow I_{in} = I_2$
- Thus via Ohm's law

$$I_{in} = \frac{V_{in}}{R_1} \qquad I_2 = -\frac{V_{out}}{R_2} \Rightarrow \text{gain} = -\frac{R_2}{R_1}$$

- So when there is negative feedback, gain is independent of A (provided V_{out} < saturation voltage)

- A non-inverting amplifier can be constructed by applying V_{in} to the +ve terminal and grounding R_1
  ↳ conserving current and applying Ohm's law gives:

$$\text{gain} = \frac{V_{out}}{V_{in}} = 1 + \frac{R_2}{R_1}$$



- By replacing R_2 with some Z_2, we can selectively amplify certain frequencies.
- In the case where $R_2/R_1 = 0$ (i.e break R_1), we have a buffer

↳ $V_{in} = V_{out}$ but $Z_{in}\uparrow$ and $Z_{out}\downarrow$
↳ i.e can connect circuits together

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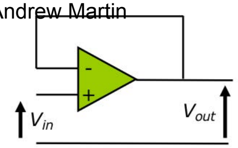
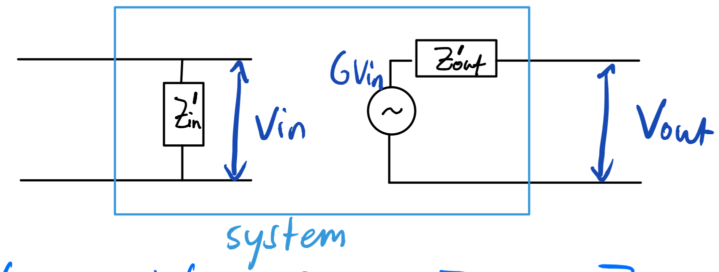


Non-ideal op-amps

- In reality:
  - A is finite $(\sim 10^4 - 10^6)$
  - $Z_{in} \neq \infty$, $Z_{out} \neq 0$
  - A displays reactance (i.e freq-dependence)
  - circuitry has a finite slew rate $\frac{dV_{out}}{dt}$
- Consider a non-ideal op-amp wired as a non-inverting amp.
  ↳ we can treat the whole circuit as a system



$$V_{out} = V_{in} \cdot G - I_{out} \cdot Z_{out}$$

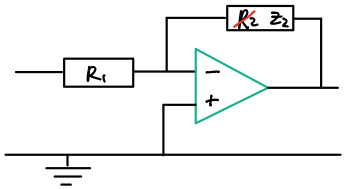
G ← open-loop gain of system

↳ this system has lower Z_{out}' than the Z_{out} of the op-amp: $Z_{out}' \rightarrow \frac{Z_{out}}{A}\left(1 + \frac{R_2}{R_1}\right)$

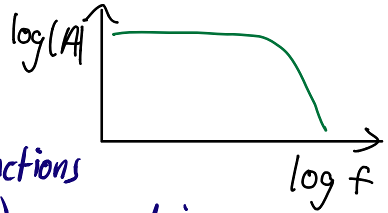
↳ and higher Z_{in}' than Z_{in}: $Z_{in}' \rightarrow \frac{Z_{in} A}{1 + R_2/R_1}$

Frequency dependence

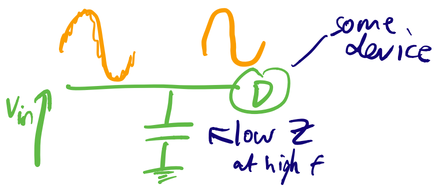
- In reality, R_2 will actually have complex impedance



- Thus at high freqs, there will be a phase shift
  - ↳ when $\phi = \pi$, the negative feedback becomes positive feedback
- Thus, it is normal that gain $\propto$ $1/\text{freq}$



- This is important because step functions (with high-freq Fourier components) may trigger positive feedback.
- To fix this, we can use decoupling capacitors:



General feedback

$$\text{Output} = A\,(\text{input} + \beta \cdot \text{output})$$

↖ feedback fraction

- Closed loop gain: $\dfrac{\text{output}}{\text{input}} = \dfrac{A}{1 - A\beta}$
- For $\beta < 0$ and $|A\beta| \gg 1$: $\text{gain} = -\dfrac{1}{\beta}$
- This independence of A is very important: as long as A is large, fluctuations / non-linearity don't matter.

- Positive feedback can be exploited, e.g if $\beta > 0$ at a given freq, that freq will dominate
  - ↳ if this was caused by a phase shift, the result will be an oscillator.

Errors

- Random error has a mean of zero. Systematic error is everything that isn't random.
- The *tolerance* is the full range within which values may lie.
- For a set of observations, the uncertainty in the mean is given by $s_m = \frac{s}{\sqrt{n-1}}$ ← sample std
- To propagate errors for $f(x, y, \ldots)$:

$$\sigma_f^2 = \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial f}{\partial y}\right)^2 \sigma_y^2 + \ldots$$

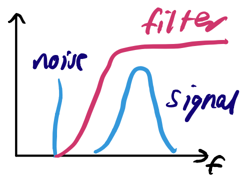
↳ if f is complicated, we can use a first order approximation: $\frac{\partial f}{\partial x} \approx \frac{f(\bar{x}+\sigma_x, \bar{y}, \ldots) - f(\bar{x}-\sigma_x, \bar{y}, \ldots)}{2\sigma_x}$

Systematic errors

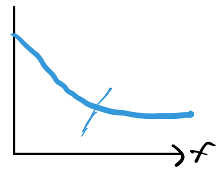
- These are all errors that are not random – can only be reduced by experiment design.
- They may change with time, eg if measurements affect the system.

1. Calibrate equipment against some known reference
2. Look for changes with time. This can be mitigated by randomising the order in which readings are taken.
3. Exploit symmetry by reversing inputs – e.g reversing polarity of input voltage to a circuit reveals a systematic error due to the electrochemical potential between contacts made of different metals.
   ↳ avoid *backlash* of mechanical instruments by approaching from the same side.
4. Use a *null method*, in which the quantity being measured is opposed by an another adjustable quantity until an indicating device shows balance.
5. Measure the difference in the quantity between two states, e.g a thermocouple with one lead in ice water.

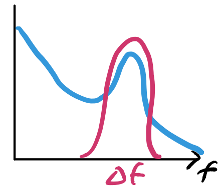
Filtering and phase-sensitive detection



- Filtering is most effective if the signal and noise have non-overlapping spectra, in which case we just need the edge to rise sufficiently fast.

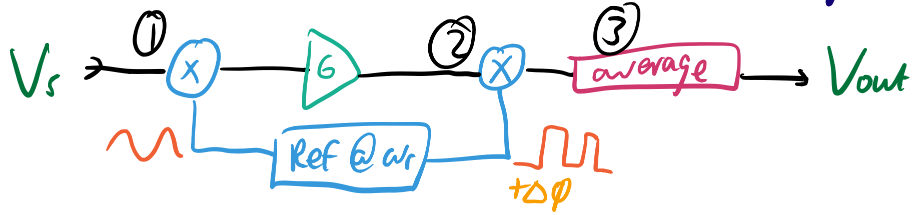


- A typical noise spectrum consists of $1/f$ noise at low frequencies, and constant *white noise* at high frequencies. The total rms fluctuation is then $\sqrt{\int P(f)\,df}$.



- For $1/f$ noise, an optimal filter would have Δf equal to the intrinsic width of the signal

- Phase-sensitive detection exploits filtering to reduce the effect of noise on the measurement of a DC signal.



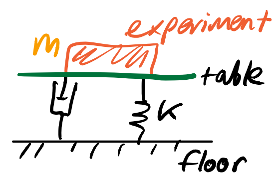
① The reference modulates the signal with a sine wave then amplifies it, resulting in $V_s \sin(\omega_r t)$ ← ignoring gain

② The reference demodulates with a square wave having a phase diff ϕ to the first.

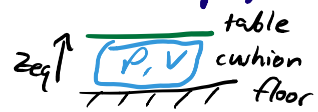
③ The output is time-averaged $\langle V_{out} \rangle = \frac{2}{\pi} V_s \cos\phi$

- Noise entering after modulation will be randomised during demodulation and will average to zero.
- For unwanted noise before modulation, with frequency $\omega_r + \Delta\omega$, it will average to zero provided that the time period of averaging is sufficiently large: $\tau \gg 1/\Delta\omega$
- Even noise at ω_r will average out if it is incoherent. Error improves as $1/\sqrt{\tau}$
- Nevertheless, best to modulate at ω_r where there is little noise. Since $1/f$ noise is normally the limiting factor, choose large ω_r.

Eliminating mechanical vibrations



- Vertical vibrations in the floor cause a forced-oscillator response in the experiment, with some resonance frequency $\omega_0^2 = k/m$
- We wish to lower ω_0 such that the resonant frequencies of the experiment $\gg \omega_0$. This can be done with damping, e.g. an air cushion of area A.



↳ assume adiabatic change $\therefore pV^\gamma = \text{const}$

↳ $F = mg = pA \Rightarrow \frac{dF}{F} = \frac{dp}{p} = -\gamma \frac{dV}{V} \sim -\gamma \frac{dz}{z_{eq}}$

↳ this provides a restoring force $dF = -\gamma \frac{dz}{z_{eq}} mg$

$\Rightarrow \boxed{\omega_0^2 = \frac{\gamma g}{z_{eq}}}$ ← result is $\omega_0 \approx 1\,\text{Hz}$, sufficiently small

Eliminating thermal noise

- Methods to reduce heat-transfer:
  - use a lid to stop evaporation
  - insulate to reduce conduction
  - put in vacuum to reduce conduction/convection
- The radiation power flux between a hot and cold surface is:

emissivity $P = \sigma\varepsilon (T_H^4 - T_c^4)$

↳ use shiny materials, with lower ε

↳ if we introduce a shiny barrier between hot and cold surfaces, the net heat flow in equilibrium is $P = \frac{1}{2}\sigma\varepsilon(T_h^4 - T_c^4)$

↳ using n floating shields reduces heat flow by a factor of $1/n+1$

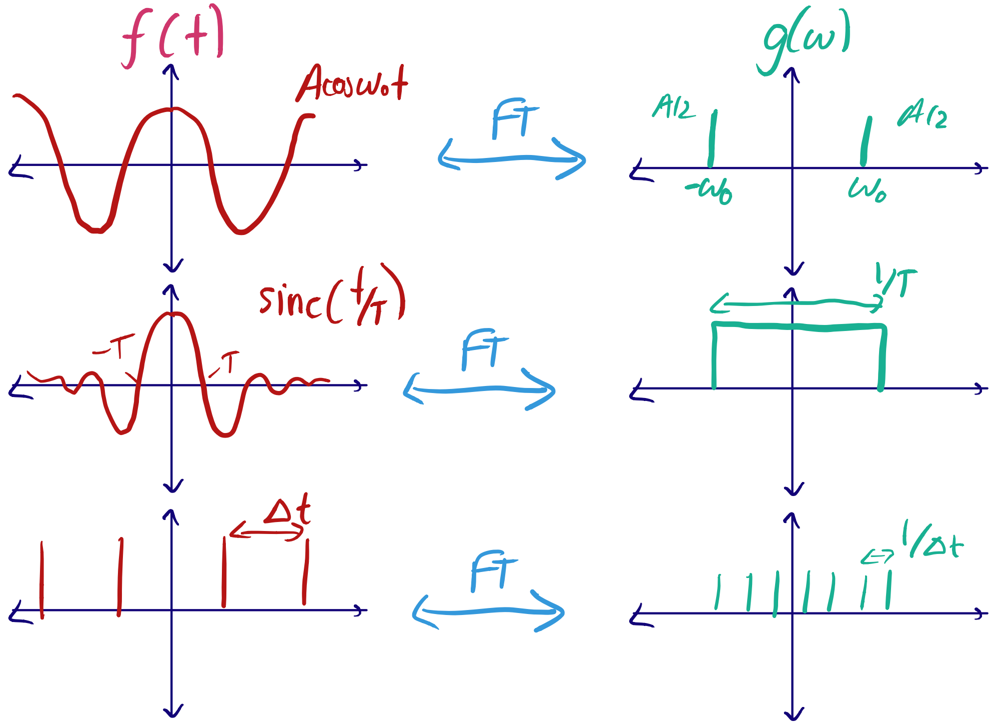
Electrical shielding

- Use a bridge circuit to compare impedances
- Use a twisted pair of wires such that roughly the same path is followed in space ⇒ same current induced.
- Use a differential amplifier to ignore all common-mode induced signals
- No large loops in the circuit to reduce pickup
- No earth loops (multiple paths to ground), since current may flow in unpredictable ways.
- A Faraday cage can be used to completely shield an electric field
- For magnetic fields, we use a shield with high μ_r to create a low reluctance path for the field lines.

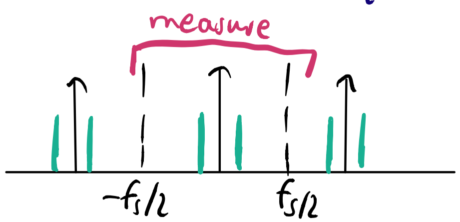
Sampling

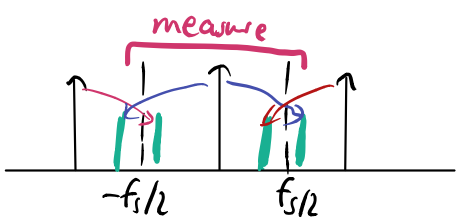
- It is possible to store analogue data (e.g vinyl), however nowadays digital sampling is preferred.
- A Fourier series represents a periodic function in the frequency domain
- An aperiodic function can be thought of as having infinite period → Fourier transform, with the spectrum becoming a continuous function in general

$$g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt \quad ; \quad f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega t}\,d\omega$$

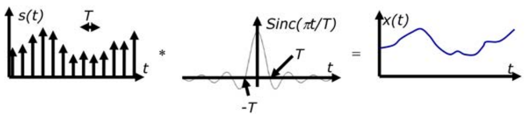


- Sampling is equivalent to multiplying some signal by a unit-impulse delta comb with spacing $\Delta T = 1/f_s$
  - ↳ f_s is the sampling frequency
  - ↳ hence the spectrum of the sampled freq is equal to the signal spectrum convolved with a delta comb.
- For a sinusoidal signal, the FT is a pair of delta spikes
  - ↳ above a signal freq of $f_s/2$, the convolution images overlap. Aliasing occurs, and we measure a freq lower than the true freq.





- This implies Nyquist's criterion:
  - for a band-limited function (ie finite-width frequency spectrum) the minimum sample freq is twice the highest freq Fourier component in the function
  - if the sampling is noiseless, the signal can be recovered perfectly.
- To recover the signal, we multiply the sample by a top-hat (to get one period) then inverse FT
  - ↳ in real space, this is the same as convolving with a sinc function.



- If we know that a signal only contains freqs in a band above 0Hz, we may achieve sub-Nyquist sampling.
  - ↳ if the bandwidth is B, the minimum sampling freq is $2B$, rather than Nyquist's $2f_{max}$.
  - ↳ need a shifted top-hat to recover the signal.

Quantisation

- Digitisation requires quantisation + sampling
- N-bit sampling means there are 2^N quantisation bins.
- For a noisy signal, we do not need to quantise/sample finely.
- Oversampling reduces quantisation error for a finite number of bins. Averaging N samples improves resolution by a factor of $\sqrt{N}$.

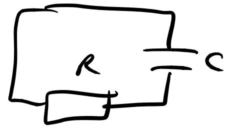
Probability and Inference

- Binomial distribution: For binary outcomes with some success probability p. $P(X=r) = \binom{n}{r} p^r (1-p)^r$
  - ↳ $E(x) = np$ $Var(x) = np(1-p)$
  - ↳ peak around np, becomes relatively narrower as n increases.
  - ↳ skewed for $p \neq 1/2$
- We can model arrival rates as a binomial dist with rare events and many trials $(p \to 0,\ n \to \infty)$
  - ↳ with λ events per interval, the probability of an event in a small subinterval is $p = \lambda/n$
  - ↳ $X \sim B(n, \lambda/n)$. In the limit, this gives the Poisson dist: $P(X=r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$
  - ↳ If $X \sim Po(\lambda)$, $E(x) = Var(x) = \lambda$
  - ↳ poisson dist is broader than binomial and has a long upper tail.
- A current composed of discrete charges arriving at random times can be modelled as a Poisson process.
  - ↳ if N electrons arrive in time Δt (on average) the current fluctuation is $\Delta I \approx \sqrt{N} \times e/\Delta t$
  - ↳ this is shot noise: $\boxed{\Delta I_{rms} \approx 2 I_{avg}\, e\, \Delta f}$

- The binomial and Poisson distributions both tend to the Gaussian for large N, λ respectively.

$$f(x \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

- The Central Limit Theorem states that the sum/mean of many independent samples (from any distribution) has a normal dist.
- Thermal noise (a.k.a Johnson noise) exhibits Gaussian statistics.
  - ↳ mean energy in atom vibration is $\frac{1}{2}kT$
  - ↳ e.g in an RC circuit, the rms thermal noise power is: $\boxed{\langle P \rangle = 4kT\Delta f}$



Parameter estimation

- We typically want to fit the parameter vector $\underline{a}$ of a model in response to new data y_i.
  - ↳ the likelihood of the dataset is the probability of seeing the dataset given the parameters:

$$L(y_1 \ldots y_N \mid \underline{a}) = \prod_i p(y_i \mid \underline{a})$$ ← assumes independence

  - ↳ the maximum-likelihood approach finds the value of $\underline{a}$ that maximises $L(y_1 \ldots y_N \mid \underline{a})$
  - ↳ but actually we want the most believable parameters given the data.
- Bayes' theorem gives:

$$\underbrace{P(\underline{a} \mid data)}_{\text{posterior}} = \frac{p(data \mid \underline{a})\, \overbrace{P(\underline{a})}^{\text{prior}}}{\underbrace{P(data)}_{\text{normalisation}}}$$

- It is important to decide on the prior:
  - no knowledge of magnitude $\Rightarrow$ uniform in log space
  - range known $\Rightarrow$ uniform prior $\leftarrow$ equivalent to max. likelihood
  - strong prior $\Rightarrow$ very strong data needed

- To fit a linear model, we vary an independent variable x_i and observe a value y_i with error σ_i. The model predicts $f(x_i|\underline{a})$. Assuming Gaussian error:

$$P(y_i|\underline{a}) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left[-(y_i - f(x_i|\underline{a}))^2/2\sigma_i^2\right]$$

$$\therefore \ln L = -\frac{1}{2}\sum_i \left[\frac{y_i - f(x_i|\underline{a})}{\sigma_i}\right]^2 - \sum_i \ln(\sigma_i\sqrt{2\pi})$$

↳ thus to maximise L we need to minimise the χ^2 statistic

$$\boxed{\chi^2 \equiv \sum_i \left[\frac{y_i - f(x_i|\underline{a})}{\sigma_i}\right]^2}$$

↳ χ^2 weighs deviations by the error σ_i^2.

- For a straight-line fit with constant error:

$$\chi^2 = \frac{1}{\sigma^2}\sum_i (y_i - mx_i - c) \Rightarrow \hat{m} = \frac{\mathrm{Cov}(x,y)}{\mathrm{Var}(x)} \qquad \hat{c} = \bar{y} - \hat{m}\bar{x}$$

↳ errors in parameter estimates are written in terms of $\hat{\sigma}^2$, which quantifies the deviation between data and model.

$$\boxed{\begin{aligned} &\hat{\sigma}^2 = \frac{1}{N-2}\sum_i (y_i - (\hat{m}x_i + \hat{c}))^2 \\ &\Rightarrow \sigma_m^2 = \frac{\hat{\sigma}^2}{N \times \mathrm{Var}(x)} \qquad \sigma_c^2 = \sigma_m^2 \frac{\sum x_i^2}{N} \end{aligned}}$$

↳ σ_m and σ_c depend on $\frac{1}{\mathrm{Var}(x)}$, hence parameter errors decrease if we explore feature space.

↳ for a straight-line fit where errors are different for each measurement, the previous formulae become weighted by $1/\sigma_i^2$ and normalised by $\sum_i 1/\sigma_i^2$.

- The χ^2 statistic can be used for hypothesis testing. If $f(x)$ truly models the data, $|y_i - f(x_i)|$ should equal σ_i on average, so $\chi^2 \approx$ d.f., where the number of degrees of freedom is $N_{data} - N_{param}$.
  - ↳ if $\chi^2 \gg$ d.f., the model is likely incorrect.
  - ↳ if $\chi^2 \ll$ d.f., we likely overestimated σ_i.