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Oscillations and Waves

Oscillations

- Consider a driven harmonic oscillator subject to damping: $m\ddot{x} + b\dot{x} + kx = F(t)$
- This can be written in the canonical form:

$$\boxed{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F}{m}} \qquad \omega_0^2 = \sqrt{k/m} \quad \gamma = b/2m$$

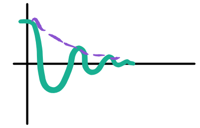
↳ we define the quality factor as the number of radians of oscillation required for energy (not amplitude!) to fall by a factor of e: $\boxed{Q = \frac{\omega_0}{2\gamma}}$

- The solution to the driven SHM equation is a linear superposition of the transient response (i.e complementary function) and the steady state (particular integral)
- With no driving force, we can easily solve the homogeneous equation $p^2 + 2\gamma p + \omega_0^2 = 0$

$$\Rightarrow p_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

↳ the relative values of γ and ω_0 determine the regime

- Light damping: $\gamma < \omega_0$, $Q > 0.5$



↳ $p = \gamma \pm i\omega_d$, $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$

$\therefore z(t) = Ae^{-\gamma t}e^{i\omega t}$ ← $A = A_0 e^{i\phi}$

$\therefore x(t) = a_0 e^{-\gamma t}\cos(\omega_d t + \phi)$ ← note: two constants

↳ here we treat SHM as the real part of a complex phasor, rotating ↺ on an Argand diagram.

↳ energy decays twice as fast as amplitude

- Heavy damping: $\gamma > \omega_0$, $Q < 0.5$

↳ resulting motion is the sum of two exponentials

$$x(t) = Ae^{-p_1 t} + Be^{-p_2 t}$$

↳ at large times, the exponential with smaller decay rate will dominate.

- Critical damping: $\gamma = \omega_0$, $Q = 0.5$

↳ most rapid approach to equilibrium, with no overshoot

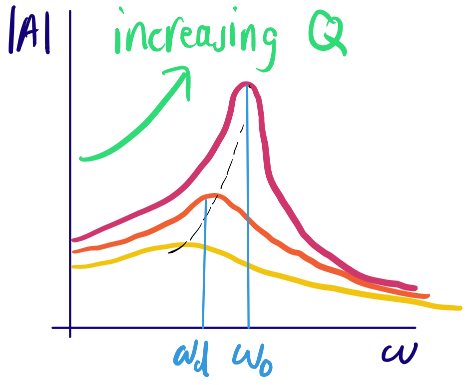
$$x(t) = (A + Bt)e^{-\gamma t}$$

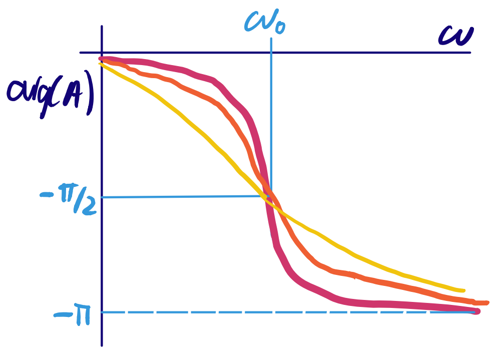
Driven harmonic oscillators

- If the forcing function is sinusoidal, i.e of the form $fe^{i\omega t}$, we use a trial solution $Ae^{i\omega t}$, where $A \equiv a_0 e^{i\phi}$. We find that

$$A = \frac{f}{\omega_0^2 - \omega^2 + 2i\gamma\omega} \quad \leftarrow f \equiv \frac{F_0}{m}$$

↳ the response function is just $Ae^{i\omega t}/F(t)$





- Response at different regimes:
  - ↳ low freq - motion controlled by spring stiffness $x = f/\omega_0^2 \cos\omega t$
  - ↳ high freq - motion controlled by inertia $x = -f/\omega^2 \cos\omega t$ (antiphase)
  - ↳ at resonance, the response is Q times larger than the $\omega \to 0$ limit

- The velocity response can be found by differentiation:
  - ↳ it has maximum value at $\omega = \omega_0$ regardless of damping
  - ↳ velocity is in phase with the driving force at resonance.
- Acceleration resonance occurs above ω_0

- The power of an oscillator can be found by multiplying the real parts of $\hat{F}$ and $\hat{v}$.

$$P = \mathrm{Re}(\hat{F})\mathrm{Re}(\hat{v}) = \tfrac{1}{2}(\hat{F} + \hat{F}^*) \cdot \tfrac{1}{2}(\hat{v} + \hat{v}^*)$$

$$\therefore \boxed{\langle P \rangle = \tfrac{1}{2}\mathrm{Re}(\hat{F}_0 \hat{v}_0^*)}$$

↳ hence, mean power depends on the phase difference between force and velocity. Maximum power when F and v are in phase.

↳ in a damped oscillator, the mean power dissipation is given by $\langle P \rangle = \frac{1}{2} b |v_0|^2$

- The width of a power resonance curve can be characterised by its half-power bandwidth

$$\omega_{hp} = \mp\gamma \pm \sqrt{\omega_0^2 + \gamma^2} \Rightarrow \Delta\omega = 2\gamma$$

↳ this provides an alternative definition for the quality factor: $\frac{\Delta\omega}{\omega_0} = \frac{1}{Q}$ i.e high Q oscillators have narrow resonance peaks.

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- Because the harmonic oscillator is a linear system, when multiple driving forces are applied we can consider each individually then sum those solutions.
- If the two driving frequencies are the same, we can use phasor analysis:

$$A^2 = A_1^2 + A_2^2 + 2A_1A_2\cos(\alpha_2 - \alpha_1)$$

↳ when two sources are coherent, the resulting power can quadruple

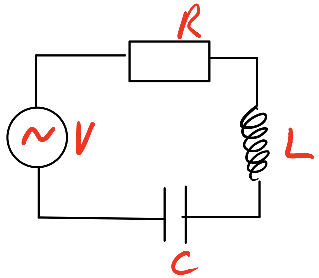
- With different driving frequencies, we see beating, with a fast oscillation at $\frac{1}{2}(\omega_1 + \omega_2)$ enveloped by a slower wave with angular frequency $\frac{1}{2}(\omega_1 - \omega_2)$.

Electrical resonance

- By Kirchoff's voltage law:

$$V_L + V_R + V_C = V(t)$$

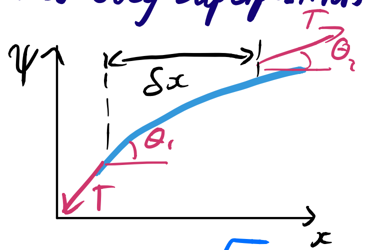
$$\Rightarrow \boxed{\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{V(t)}{L}}$$



- This is clearly SHM with $\omega_0^2 = \frac{1}{LC}$, $\gamma = \frac{R}{2L}$
  - ↳ inductance ⟷ mass, resistance ⟷ damping, 1/capacitance ⟷ spring constant.
  - ↳ for RLC circuits, the quality factor is: $Q = \frac{1}{R}\sqrt{\frac{L}{C}}$
  - ↳ current is greatest when $\omega = 1/\sqrt{LC}$ (velocity resonance)
  - ↳ dissipated power is given by $\langle P \rangle = \frac{1}{2}\mathrm{Re}[\,V_0 \hat{I}_0^*\,]$

Waves

- A wave is described by $\Psi(x,t) = f(x \pm ct)$ (−ve for +x travel)
  - ↳ by taking partial derivatives, we can derive the 1D wave equation: $\frac{\partial^2\Psi}{\partial t^2} = c^2\frac{\partial^2\Psi}{\partial x^2}$
  - ↳ if c is constant, the wave is non-dispersive
  - ↳ because the equation is linear, waves obey superposition
- Eg a string under tension experiences a restoring force towards the axis:



$$F = T\left(\frac{\partial\Psi}{\partial x}\Big|_x - \frac{\partial\Psi}{\partial x}\Big|_{x+\delta x}\right) = -T\frac{\partial^2\Psi}{\partial x^2}\delta x$$

↳ By NII, $\frac{\partial^2\Psi}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2\Psi}{\partial x^2}$, hence $c = \sqrt{\frac{T}{\rho}}$

- A harmonic wave has a displacement that varies sinusoidally with time at any x.

$$\Psi(x,t) = \mathrm{Re}\left\{Ae^{i(\omega t - kx)}\right\}$$

↳ k is the wavenumber: $k = 2\pi/\lambda$

- The general wave equation is $\boxed{\frac{\partial^2\Psi}{\partial t^2} = c^2\nabla\Psi}$
- In 3D, a harmonic plane wave is described by:

$$\Psi(\underline{r},t) = \mathrm{Re}\left\{A\exp(i(\omega t - \underline{k}\cdot\underline{r}))\right\}$$

↳ the wavenumber becomes a wavevector
↳ by taking $\partial/\partial t$ (i.e multiplying by $i\omega$) and the grad (i.e multiplying by $-i\underline{k}$), we can show that:

$$c^2 = \frac{\omega^2}{|\underline{k}|^2}$$

- A spherical wave does not vary with θ, ϕ. We can show by substitution that a valid solution is:

$$\psi(r,t) = \frac{f(r \pm ct)}{r}, \quad \text{e.g} \quad \hat{\psi}(r,t) = \frac{Ae^{i(\omega t - kr)}}{r}$$

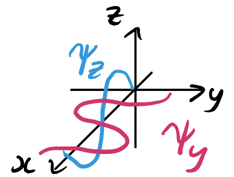
↳ the $1/r$ dependence is consistent with the inverse square law for power.

- A cylindrical wave can be generated from a line source (e.g diffraction slit)
↳ we may guess a $1/\sqrt{r}$ dependence to conserve power:

$$\psi(r,t) = \frac{f(r \pm ct)}{\sqrt{r}}$$

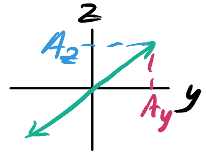
↳ this is not a solution, but is a good approx for $r \gg \lambda$ (far away from slit).

Polarisation



- A transverse wave can be disturbed along two axes (because of superposition).
↳ the relative amplitudes and phases define the polarisation
- In general:

$$\psi_y = A_y \cos(\omega t - kx)$$
$$\psi_z = A_z \cos(\omega t - kx + \phi)$$



- Linear polarisation arises when $\phi = 0$
↳ any linearly polarised wave can be resolved int two orthogonal components with the same phase.
- Circular polarisation occurs when $A_y = A_z$ but $\phi = (m + \frac{1}{2})\pi$
↳ displacement vector traces a corkscrew
- The general case is elliptical polarisation:
↳ two amplitudes and an angle are needed to specify
↳ waves can be partially polarised – in which case another parameter specifies the unpolarised power.
- Polarised waves can be represented with 2-vectors (y and z components):
  - e.g linearly polarised $\begin{pmatrix} A \\ 0 \end{pmatrix}$ – along y $\begin{pmatrix} 0 \\ A \end{pmatrix}$ – along z
  - e.g circular $\begin{pmatrix} Ai \\ A \end{pmatrix}$

Reflection and transmission

- Wave impedance relates the transverse force to the transverse velocity (NOT wave velocity)

$$\text{impedance} = \frac{\text{driving force}}{\text{transverse velocity}}$$

↳ for a string: $Z = \rho c = \sqrt{T\mu}$

- For a wave with transverse velocity $\hat{u}$:

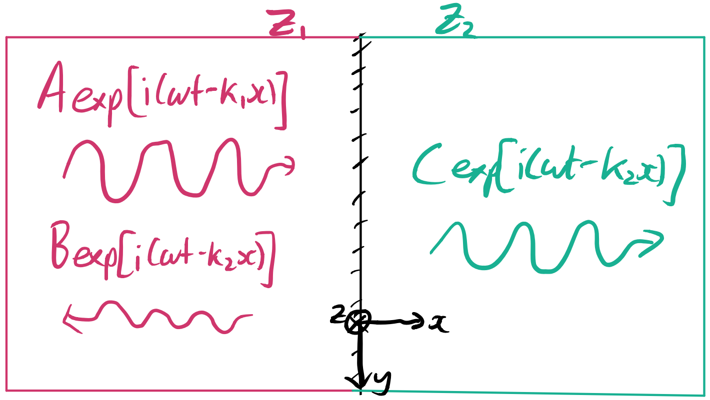
$$\langle P \rangle = \tfrac{1}{2}\,\mathrm{Re}(\hat{F}\hat{u}^*) = \tfrac{1}{2}\,\mathrm{Re}(\hat{Z})\,|\hat{u}|^2$$

↳ if Z is real and the wave is harmonic:

$$\langle P \rangle = \tfrac{1}{2} Z \omega^2 A_0^2$$

↳ this can also be derived by considering KE and PE per unit length: $KE = \frac{1}{2}\rho\left(\frac{\partial\psi}{\partial t}\right)^2$ $\quad PE = \frac{1}{2}T\left(\frac{\partial\psi}{\partial x}\right)^2$

- Consider a harmonic wave approaching a boundary:



↳ B.Cs at $x=0$: ψ is continuous

$\frac{\partial\psi}{\partial x}$ is continuous ← related to transverse force

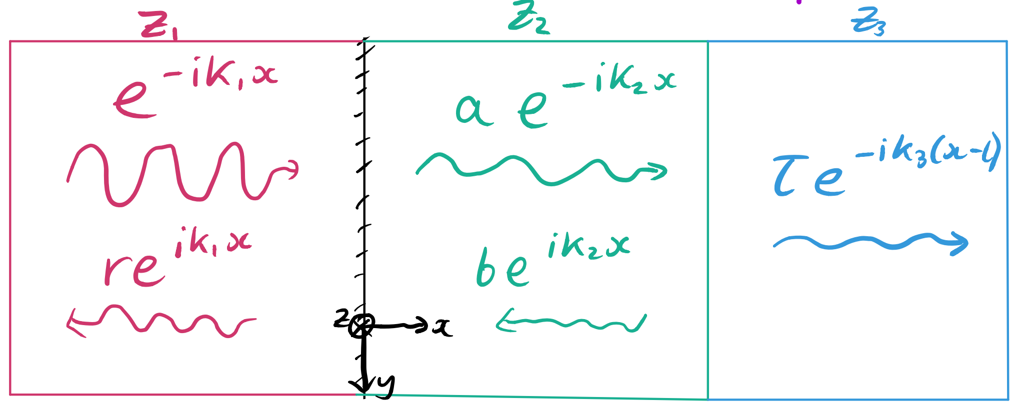
$\therefore\ A + B = C$ and $Z_1(A-B) = Z_2 C$

↳ the reflection coefficient and transmission coefficient:

$$r = \frac{B}{A} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \qquad \tau = 1 + r = \frac{2Z_1}{Z_1 + Z_2}$$

- The power coefficients are found by squaring τ and r (technically square modulus).

- To reduce reflections at interfaces, we can use impedance matching



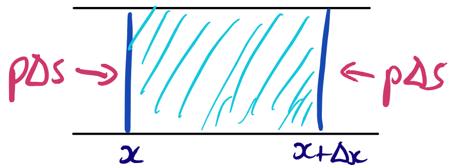
↳ to simplify algebra: drop $e^{i\omega t}$, set incident amplitude to 1, add a phase shift $e^{ik_3 l}$ to the transmission, and define $\gamma = e^{-ik_2 l}$

↳ match boundary conditions to find r, a, b, τ

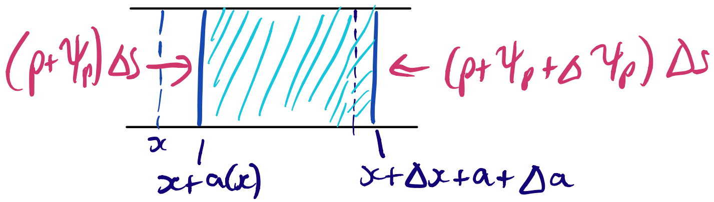
- If we choose a quarter wavelength of material, the reflected waves from each boundary are out of phase.
  - ↳ we also note that the effective impedance of the layer and substrate is given by $Z_{eff} = Z_2^2 / Z_3$
  - ↳ so to match impedances, we need $Z_2 = \sqrt{Z_1 Z_3}$
  - ↳ in practice, Z for a material can be found via $Z = \frac{Z_0}{n}$

Longitudinal waves

- Longitudinal waves displace the medium in the same direction as they propagate ⇒ no polarisation.
- Sound waves propagate by compressions and rarefactions of a medium (caused by pressure waves)
- We analyse an infinitesimal column of gas with area ΔS and equilibrium pressure p.



No wave



wave causes displacement $a(x)$ and pressure ψ_p

↳ the fractional change in the column volume caused by the wave is $\frac{\Delta V}{V} = \frac{\Delta S \frac{\partial a}{\partial x} \Delta x}{\Delta S \Delta x} = \frac{\partial a}{\partial x}$

↳ similarly $F_{net} = \Delta P \Delta S = -\frac{\partial \psi_p}{\partial x} \Delta x \Delta S$

- Because the pressure changes rapidly, there is no time for heat exchange ⇒ adiabatic process, $pV^\gamma = \text{const}$

$$\Rightarrow dp = \gamma p \frac{\Delta V}{V} = -\gamma p \frac{\partial a}{\partial x}$$

↳ dp is the pressure change from the wave, i.e $dp \equiv \Psi_p$

$$\therefore \frac{\partial \Psi_p}{\partial x} = -\gamma p \frac{\partial^2 a}{\partial x^2} - \gamma \frac{\partial p}{\partial x}\frac{\partial a}{\partial x}$$

↳ ratio of 2nd/1st terms on RHS $\sim a/\lambda$ so is negligible. Hence $F_{net} \propto \frac{\partial^2 a}{\partial x^2}$

- By NII, $F_{net} = \rho \Delta x \Delta S \ddot{a}$

$$\Rightarrow \frac{\partial^2 a}{\partial t^2} = \frac{\gamma p}{\rho}\frac{\partial^2 a}{\partial x^2}$$

↳ nondispersive wave with $\boxed{c = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma RT}{M}}}$ molar mass

- It is easier to measure changes in pressure:

$$\Psi_p = -\gamma p \frac{\partial a}{\partial x} \ \& \ a = a_0 e^{i(\omega t - kx)} \Rightarrow \boxed{\Psi_p = i\gamma p k a}$$

↳ i.e pressure leads displacement by $\pi/2$

↳ the acoustic impedance $\mathcal{L}$ is the impedance per unit area:

$$\mathcal{L} = \frac{\text{force}}{\text{velocity} \times \text{area}} = \frac{\Psi_p \Delta S}{\dot{a}\Delta S} = \frac{i\gamma p k a}{i\omega a} = \boxed{V\rho = \frac{\gamma p}{V}}$$

- The intensity of a wave is the mean power per unit area

$$I = \frac{1}{2}\mathrm{Re}[\Psi_p \dot{a}^*] = \frac{1}{2}\mathcal{L}\omega^2|a_0|^2 = \frac{|A_0|^2}{2\mathcal{L}}$$

$a = a_0 e^{i(\omega t - kx)}$ $\qquad$ $\Psi_p = \hat{A}_0 e^{i(\omega t - kx)}$

- The decibel scale is a logarithmic relative scale:

↳ sound pressure level $= 20\log_{10}(p_{rms}/p_{ref})$

↳ $p_{ref} = 20\mu\,\mathrm{Pa}$, roughly the threshold of human hearing

↳ alternatively: $dBA = 10\log_{10}(I/I_{ref})$, $I_{ref} = 10^{-12}\,\mathrm{Wm^{-2}}$

- Longitudinal waves also occur in liquids and solids. The derivation is similar except for the relationship between pressure and volume. In general:

$$dp = \Psi_p = -K\frac{\partial a}{\partial x} \quad \leftarrow K \text{ is the elastic modulus}$$

↳ the wave speed is then $c = \sqrt{\frac{K}{\rho}}$

↳ for gases and liquids we use the bulk modulus since pressure is isotropic: $dp = -B\frac{dV}{V}$

↳ solids are more complex because of shear stresses and Poisson's ratio. However, for thin bars we can just use Young's modulus $\sigma = Y\frac{\partial a}{\partial x} \Rightarrow c = \sqrt{Y/\rho}$

Standing waves

- Standing waves form from the superposition of forward/backward waves with some B.C. $\Psi(x,t) = X(x)T(t)$.
- e.g for a string of length l with $\Psi(0,t) = \Psi(l,t) = 0$,

$$\Psi = A\cos(\omega t - kx) - A\cos(\omega t + kx) = 2A\sin\omega t \sin kx.$$

↳ B.Cs satisfied when $k = n\pi/l$, $n \in \mathbb{Z}^+$

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Damped waves

- Assume a damping force ∝ transverse speed

$$\therefore \frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2} - \Gamma \frac{\partial \psi}{\partial t}$$ $\Gamma \equiv \beta/\mu$, β is the damping const

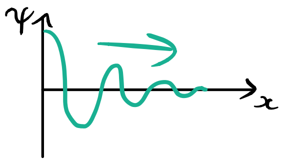
↳ if we try harmonic waves, k must be complex

$$k = k_r - i k_i \Rightarrow \boxed{k_r^2 - k_i^2 = \frac{\omega^2}{c^2}; \quad 2 k_r k_i = \frac{\Gamma \omega}{c^2}}$$

- For light damping $\Gamma \ll \omega$ so $k_r \approx \frac{\omega}{c}$, $k_i \approx \frac{\Gamma}{2c}$

$$\therefore \psi(x,t) = e^{-k_i x} \mathrm{Re}\left[D e^{i(\omega t - k_r x)}\right]$$

↳ decaying travelling wave
↳ 'damping length' set by k_i, and independent of wavelength.



- For heavy damping $\Gamma \gg \omega \Rightarrow -i\Gamma\omega \approx c^2 k^2$

$$\therefore k' = k_r \approx k_i \approx \pm \sqrt{\frac{\Gamma \omega}{2c^2}}$$ ← real and imag parts are equal

↳ wave decays over a short distance since decay length varies as $\omega^{-0.5}$.

- The impedance of the wave now has frequency dependence:

$$Z = \frac{-T\, \partial\psi/\partial x}{\partial \psi/\partial t} = \frac{Tk}{\omega} = T/\omega\,(k_r - i k_i)$$

↳ light damping: $Z(\omega) = Z_0\left(1 - \frac{i\Gamma}{2\omega}\right)$
↳ heavy damping: $Z(\omega) = Z_0 (1-i)\sqrt{\frac{\Gamma}{2\omega}}$

- For a boundary between two (possibly damped) media, we can use the same reflection coefficient
  ↳ for light damping, $r(\omega) = \frac{i\Gamma}{4\omega}$, i.e little reflection
  ↳ for heavy damping, $r(\omega) \approx -1$, i.e antiphase reflection

- The dispersion relation is the relationship between ω and k. For non-dispersive systems, $\omega = ck$.
- For a lightly damped wave, the propagating wave has phase $\omega t - k_r x$ so the phase speed is

$$V_\phi = \frac{\omega}{k_r} = c\left(1 + \frac{\Gamma^2}{4c^2 k_r^2}\right)^{-1/2}$$

↳ wave speed now depends on wavelength so this wave is dispersive.

- Dispersion can occur without damping, e.g a stiff string that resists bending: $\frac{\partial^2 \psi}{\partial t^2} = c^2\left(\frac{\partial^2 \psi}{\partial x^2} - \alpha \frac{\partial^4 \psi}{\partial x^4}\right)$
  ↳ the harmonic solution has $\omega = \pm ck\sqrt{1 + \alpha k^2}$
  ↳ there is no loss of energy, but low wavelength waves are more affected by the stiffness (faster wave)
- This is relevant for piano tuning. Because we have $V_\phi(\lambda)$, $f_0 = V_\phi(2L)/2L$, $f_1 = V_\phi(L)/L \Rightarrow f_1/f_0 > 2$. In practice, we will thus tune the higher octave string to match f_1 instead of $2f_0$ to prevent beats.

Group velocity

- Consider two equal-amplitude waves with slightly different frequencies propagating together.

$$\psi = \cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x), \quad \omega_1 \approx \omega_2, \quad k_1 \approx k_2$$

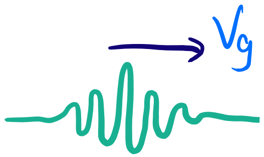
$$\Rightarrow \psi = 2\cos(\omega_+ t - k_+ x)\cos(\omega_- t - k_- x)$$

where $\omega_+ = \frac{1}{2}(\omega_1 + \omega_2)$ $\quad \omega_- = \frac{1}{2}(\omega_1 - \omega_2)$

↳ thus there is a high frequency wave with speed $v_\phi \approx \omega_1/k_1 \approx \omega_2/k_2$ modulated by a lower frequency envelope with group velocity:

$$v_g = \frac{\omega_-}{k_-} = \frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\omega}{dk}$$

- Alternatively, we can consider the speed of the 'peak' of a group. At the peak, all components add in phase, hence $(\omega t - kx + \phi)$ is constant for all components



$$\therefore \frac{d}{d\omega}(\omega t - kx + \phi) = 0 \Rightarrow \frac{x}{t} = \boxed{v_g = \left(\frac{d\omega}{dk}\right)_{\omega_0}}$$

- For a nondispersive wave, $\frac{\omega}{k} = \frac{d\omega}{dk}$ for all ω so the group maintains its shape. For dispersive waves, crests may move relative to the envelope.
- v_g is important because it is the rate of information propagation.
- The range of wavevectors in a group is inversely related to the spatial extent of the group. $\Delta k \Delta x \approx 1$

- If the group contains wavevectors in the band $k_0 \pm \Delta k$, the max and min velocities in the group are

$$v_{min} = \frac{\partial \omega}{\partial k}\Big|_{k_0 - \Delta k} \quad v_{max} = \frac{\partial \omega}{\partial k}\Big|_{k_0 + \Delta k}$$

↳ in a time t, the wave spreads by:

$$\Delta x \approx \Delta x_0 + (v_{max} - v_{min})t$$
$$\approx \Delta x_0 + 2\frac{\partial^2 \omega}{\partial k^2}\Big|_{k_0} \Delta k\, t$$

$$\therefore \boxed{\Delta x \approx \Delta x_0 + 2\frac{\partial^2 \omega}{\partial k^2}\Big|_{k_0} \frac{t}{\Delta x_0}} \leftarrow \text{using: } \Delta k \Delta x \approx 1$$

Water waves

- Water waves are complex because they have both longitudinal and transverse propagation (in quadrature).
- For deep water, we can model the dispersion relation as:

$$\omega^2 = \underbrace{gk}_{\text{gravity}} + \underbrace{\sigma k^3/\rho}_{\text{surface tension}}$$

- Ripples are surface tension driven:

$$\omega \approx \sqrt{\frac{\sigma k^3}{\rho}} \Rightarrow v_g \approx \frac{3}{2}\sqrt{\frac{\sigma k}{\rho}} = \frac{3}{2}v_\phi$$

↳ anomalous dispersion because speed ↓ as λ ↑

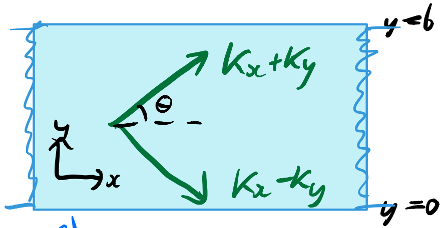
- Gravity waves have longer wavelengths and are inertia-driven.

$$\omega = \sqrt{gk} \Rightarrow v_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}v_\phi$$

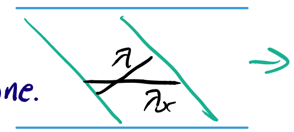
↳ normal dispersion since speed ↑ as λ ↑

- If λ exceeds the water depth h, gravity waves are mainly longitudinal and have a dispersion relationship:
$$\omega^2 = ghk^2\left(1 - \frac{h^2k^2}{3}\right)$$
  ↳ For very shallow water, $\omega \approx \sqrt{gh}$ so the waves are approximately nondispersive.
  ↳ thus when appoaching the shore, since speed ↓, amplitude ↑ to conserve water

Guided waves



- Consider a wave travelling in $+x$ along a 2D membrane
- We consider these waves as having $\underset{\sim}{k} = (k_x, \pm k_y) = (k\cos\theta, \pm k\sin\theta)$
- The total displacement is $\Psi = Ae^{i(\omega t - k_x x)}\left[e^{-ik_y y} - e^{ik_y y}\right]$
$$\therefore \Psi = -2iA\sin(k_y y)\exp[i(\omega t - k_x x)]$$
  ↳ i.e travelling wave in $+x$ with wavevector k_x modulated by a standing wave with $k_y = \frac{n y}{b}$
$$\therefore \omega^2 = c^2|\underset{\sim}{k}|^2 = c^2\left(k_x^2 + \frac{m^2\pi^2}{b^2}\right) \qquad m \in \mathbb{Z}^+$$
  ↳ hence the guided waves are dispersive
$$\therefore V_g = \frac{d\omega}{dk_x} = \frac{c^2}{\omega}\sqrt{\frac{\omega^2}{c^2} - \frac{m^2\pi^2}{b^2}}$$
- Thus the dispersion relation and displacement pattern (waveguide mode) is specified by m.



- $k_x < |k|$, so the wavelength of the unguided wave exceeds that of the guided one.
- Hence the phase velocity is greater than the wave speed, but group velocity is smaller.
- As $k_x \to 0$, $V_\phi \to \infty$ and $V_g \to 0$. $V_\phi \to \infty$ does not violate relativity since the group carries the info.
- As $k_x \to \infty$, the behaviour approaches an unguided wave.
- Below the cutoff angular frequency $\omega_c = \frac{m\pi c}{b}$, k_x^2 becomes negative so there is no propagation.
- If there is a spread of frequencies, multiple modes can be excited, resulting in signal distortion.
  ↳ avoided by choosing b such that ω is below the cutoff freq for mode $m=2$
  ↳ the guide is then single-moded for ω.

- In an optical fibre, data is transmitted via pulses of light.
  ↳ choose λ with minimal dispersion, but also minimal absorption into the fibre.
  ↳ the silica core is very thin so only one mode exits
  ↳ there is dispersion because it's a waveguide, but also because the refractive index depends on λ. Materials are chosen such that these effects cancel.

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Fourier Series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi n t}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi n t}{T}\right)dt$$

- For a square wave: $f(t) = \frac{4}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \ldots\right)$
  - ↳ first few components can be used to approx. response since it drops off rapidly at higher freqs.
- It is sometimes simpler to use a complex representation:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \qquad C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-in\omega_0 t} dt$$

- In the limit, this leads to the Fourier transform.

$$\mathcal{F}[f(t)] = g(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\mathcal{F}^{-1}[g(\omega)] = f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$$

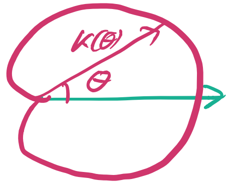
- By definition, if a linear system is driven with $f(t)$, the response (in freq domain) is: $R(\omega)\mathcal{F}[f(t)]$
  - ↳ thus the response (time) is: $x(t) = \mathcal{F}^{-1}[R(\omega)\mathcal{F}[f(t)]]$

- A single pulse at the origin (i.e a delta function) has a constant F.T. i.e it is a mix of all frequencies.
- The convolution of $f(x)$ with a delta function replicates f, centered at the delta spike.
  - ↳ by imagining another function $g(x)$ to be an infinite number of spikes with different heights, we see that $f * g$ causes g to be 'smeared out' by f.
- Hence if we know the convolution function for a noisy image, we can use deconvolution.
- If we know how the system responds to a delta function impulse, by linearity we can extend this to any driving force by modelling that force as many delta spikes
  - ↳ essentially the same method as Green's functions

- Useful rules for Fourier transforms:
  - ↳ reciprocity $\mathcal{F}[f(t)] = g(\omega) \Rightarrow \mathcal{F}^{-1}[f(\omega)] = g(-t)$
  - ↳ scaling $\mathcal{F}[f(t/a)] = |a|g(a\omega)$
  - ↳ linearity
  - ↳ convolution theorem
  - ↳ FT of real function has Hermitian symmetry, i.e $\tilde{f}(-\omega) = \tilde{f}(\omega)^*$
  - ↳ if $f(t)$ is real symmetric, so is the FT.

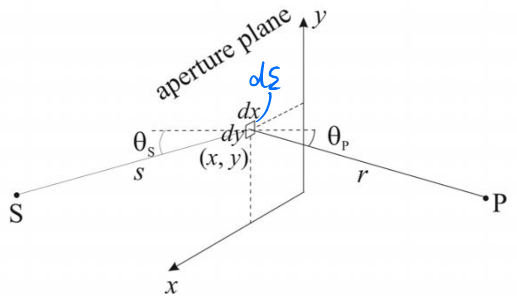
Diffraction

- Huygens' principle can be used to derive some phenomena, but it predicts a backwards-propagating wavefront.
  - ↳ to fix this, we use Huygens-Fresnel theory, introducing an inclination factor $K(\theta)$ which describes the dropoff in intensity as a function of angle.
  - ↳ Fresnel proposed $K(\theta) = \frac{1+\cos\theta}{2}$



  - ↳ the relative amplitude of the secondary wavelets is $-i/\lambda$

- Consider a planar aperture Σ, with an element (dx, dy) located at $(x, y, 0)$



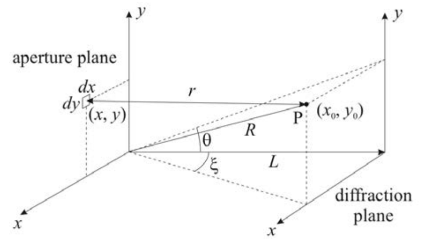
- Consider monochromatic spherical waves from S, i.e:
$$\Psi(\underline{r}, t) = Re\{\hat{\Psi}(\underline{r})e^{-i\omega t}\}$$
  - ↳ the wave arriving at $d\Sigma$ is then $\hat{\Psi}_1(\underline{r}) = \frac{a_s e^{iks}}{s}$

- The aperture can change the amplitude and phase, as characterised by a complex aperture function $\hat{h}(x,y)$
  - ↳ then the secondary wavelets are described by
$$a_\Sigma = \hat{A}\,\hat{\Psi}_1(x,y)\,\hat{h}(x,y)\,dxdy$$
relative amplitude of secondary waves
$$\therefore d\Psi_p = -\frac{i}{\lambda}\frac{a_s e^{iks}}{s}\hat{h}(x,y)\,dxdy\,K(\theta)\frac{e^{ikr}}{r}$$
  - ↳ the obliquity is given by $K = \frac{1}{2}[\cos(\theta_s) + \cos(\theta_p)]$
$$\Rightarrow \boxed{\Psi_p = \iint_\Sigma -\frac{i}{\lambda}\hat{h}(x,y)\,K(\theta_s,\theta_p)\frac{a_s e^{ik(s+r)}}{sr}\,dxdy}$$

- The diffraction integral allows us to calculate Ψ_p relatively near the aperture, but it still breaks down for $r < \lambda$, i.e the 'very near-field' case.

Fraunhofer Diffraction

- Consider the diffraction pattern on a plane, with planar waves incident normally at the aperture



$$r^2 = L^2 + (x_0 - x)^2 + (y_0 - y)^2 \Rightarrow r \approx R - \frac{x_0 x + y_0 y}{R} + \underbrace{\frac{x^2+y^2}{2R}}_{\text{negligible for large } R}$$
use binomial expansion

- ↳ specifically, we can ignore the second term if $\frac{k(x^2+y^2)}{2R} \ll \pi \Rightarrow R \gg \frac{D^2}{\lambda}$ ← max extent of aperture

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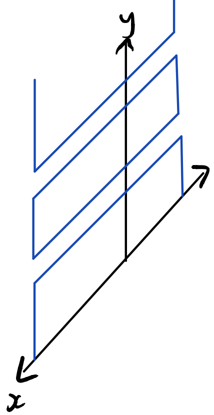
- With the approximation that $K(\theta) \simeq 1$, and using Ψ_Σ as the incident wave (constant for plane wave), we have the Fraunhofer integral:

$$\boxed{\Psi_P \propto \iint_\Sigma \Psi_\Sigma \, \hat{h}(x,y) \exp\left[-ik\left(\frac{x_0 x + y_0 y}{R}\right)\right] dx dy}$$

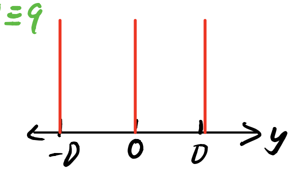
- For 1D diffraction, with patterns extending in $-\infty < x < \infty$, the integral over x is just a multiplicative constant. Using a small angle approx $\sin\theta \simeq y_0/R$:



$$\Psi_P \propto \int h(y)\, e^{-iky\sin\theta} dy$$

$$\Rightarrow \boxed{\Psi_P(\underbrace{k\sin\theta}_{\text{we write } k\sin\theta \equiv q}) \propto \mathcal{F}\{h(y)\}}$$

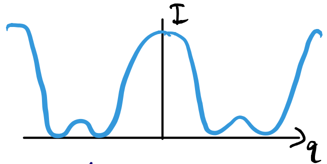
- e.g. for 3 narrow slits



$$h(y) = \delta(y+D) + \delta(y) + \delta(y-D)$$

$$\therefore \Psi_P \propto e^{iqD} + 1 + e^{-iqD} = 1 + 2\cos(qD)$$

$$\therefore I_P(q) = I_0(1 + 2\cos(qD))^2$$



↳ we can extend this to N narrow slits, resulting in:

$$\boxed{I_P = I_0 \,\text{sinc}^2(NqD/2)}$$

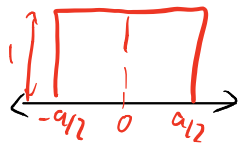
↳ As $N \to \infty$, the diffraction pattern tends to a delta comb

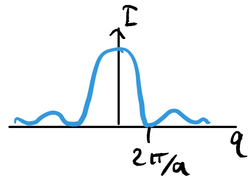
↳ the separation of primary maxima is $G = 2\pi/D$

↳ $N-2$ subsidiary maxima and $N-1$ zeroes.

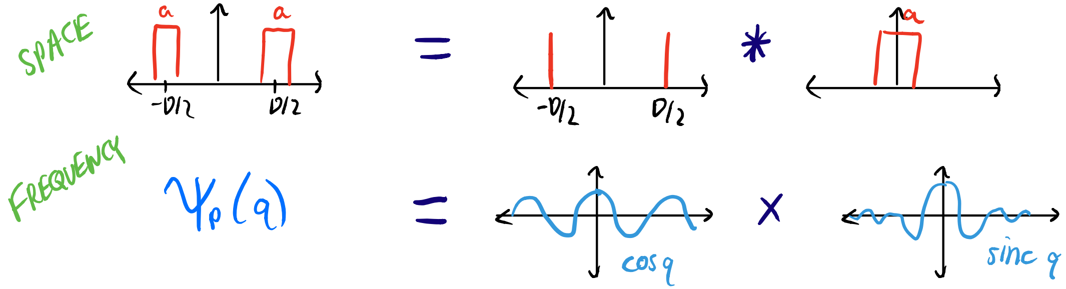
- e.g. for a wide aperture:

$$I_P(q) \propto a^2 \,\text{sinc}^2\left(\frac{qa}{2}\right)$$





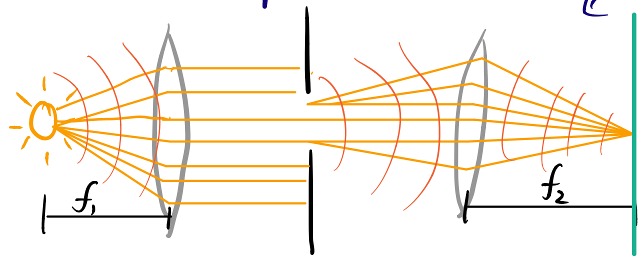
- More complicated diffraction patterns can be analysed with the convolution theorem:



$$\therefore I_P(q) = I_0 \cos^2\left(\frac{qD}{2}\right) a^2 \,\text{sinc}^2\left(\frac{qa}{2}\right)$$

↳ this modulation may lead to missing orders where a peak is expected due to a minimum in the envelope.

- If we introduce some phase-shift at the aperture, the diffraction pattern shifts.
- In practice, to make use of Fraunhofer diffraction we can use lenses to ensure that plane waves are coming in/out of the aperture



waves leaving the aperture at angle θ are focused to a point at $f_2\sin\theta$ on the screen.

- Fraunhofer diffraction can also be used for 2D apertures, using $\sin\theta \approx y_0/R$ and $\sin\xi \approx \frac{x_0}{r}$, with $q = k\sin\theta$ and $p = k\sin\xi$. This gives the 2D Fourier Transform:

$$\therefore \quad \psi_p(p,q) \propto \iint_{\Sigma} \hat{h}(x,y)\, e^{-i(px+qy)}\, dx\,dy$$

↳ this is easy to evaluate if $\hat{h}(x,y)$ is separable into $\hat{f}(x)\,\hat{g}(y)$ – then it is the product of two 1D FTs.

- A circular aperture is not separable in x, y. The Fraunhofer integral evaluates to:

$$\psi_p(q) \propto \frac{\psi_0 d^2}{2}\, \frac{J_1(qd/2)}{qd/2}$$

1st order Bessel Function of the first kind

↳ the diffraction pattern has its first zero at $\boxed{\sin\theta = \frac{1.22\lambda}{d}}$

↳ the region inside the first zero is the Airy disc, containing 86% of the energy flux.

- Babinet's Principle states that the diffracted intensities of an aperture and its complement are the same, except for the undiffracted beam

$$\psi_1 \propto \iint_A e^{-i(px+qy)}\, dx\,dy$$

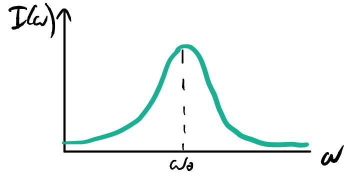
$$\psi_2 \propto \iint_{\text{all space}} e^{-i(px+qy)}\, dx\,dy - \iint_A e^{-i(px+qy)}\, dx\,dy$$

$$\therefore \quad \boxed{\psi_2 \propto \delta(p,q) - \psi_1}$$

Spectral line emission

- Spectral lines arise from transitions between quantum states.
- They have a finite width because there is a small uncertainty in their energy, because a quantum state has some lifetime.
- The electric field decays as $E(t) = E_0 e^{-\gamma t}\cos\omega_0 t$

$$\therefore \quad I(\omega) \propto \frac{1}{(\omega-\omega_0)^2 - \gamma^2}$$



↳ Lorentzian power spectrum

- Particle collisions limit the coherence of emitted waves.

↳ mean collision time depends on the number density of particles, collision cross section Σ, and v_{rms}

$$\tau_c \sim \frac{1}{n\Sigma v_{rms}} \Rightarrow \Delta\omega \sim n\Sigma v_{rms}$$

- Because the atom will be moving when it emits light, there is a Doppler shift: $\omega \approx \omega_0\left(1 + \frac{u_x}{c}\right)$ ω_0 is the rest-frame freq.

↳ hence a signal component with freq ω came from an atom with speed $u_x \approx c(\omega-\omega_0)/\omega_0$

↳ the 1D Boltzmann distribution gives:

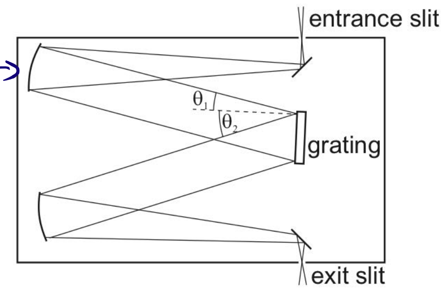
$$p(u_x) \propto \exp\left(\frac{-mu_x^2}{2k_BT}\right) \Rightarrow \boxed{I(\omega) \propto \exp\left(-\frac{mc^2(\omega-\omega_0)^2}{2\omega_0^2 k_B T}\right)}$$

↳ hence the spectrum is Gaussian. This may be dominant at higher altitudes (less atmospheric pressure broadening).

- Generally, spectra will be the convolution of Loentzian/Gaussian.

- Spectra are normally measured using grating spectrometers.

concave mirrors have less chromatic aberration than lenses



↳ a concave mirror reflects focused incident light onto a diffraction grating at a specific angle

↳ light is then diffracted according to:

$$D(\sin\theta_2 - \sin\theta_1) = m\lambda$$

grating equation with non-normal incidence.

order of maximum

Resolution

- For a diffraction grating of finite width, the intensity peaks will be finite-width peaks $\propto \mathrm{sinc}^2(Nq\,D/2)$
  - ↳ for illumination at two wavelengths (normal incidence), there will be peaks at

    $$D\sin\theta_\lambda = m\lambda \qquad D\sin\theta_{\lambda'} = m(\lambda + \delta\lambda)$$
  - ↳ the first minimum for the mth primary maximum for the λ pattern is at $D\sin\theta_\lambda = m\lambda + \frac{\lambda}{N-1}$
- The Rayleigh criterion states that the peaks will be resolved if the maximum of one pattern coincides with the minimum of the other.

- Define $R = \frac{\lambda}{\delta\lambda}$ as the chromatic resolving power of the grating:

  $$R = \frac{\lambda}{\delta\lambda} = mN$$ ← number of slits

  ↳ hence it is easier to distinguish higher-order peaks.

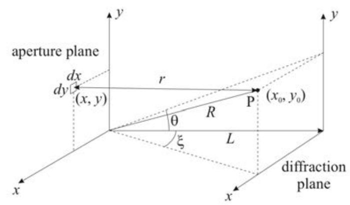
- In geometrical optics, lenses produce point images from point objects. But in physical optics, the finite circular extent of the lens produces an Airy disc.
  - ↳ the angular radius of the disc is $\alpha \approx \frac{1.22\lambda}{D}$
  - ↳ the actual radius is $\frac{1.22\lambda}{D} f$ ← focal length
  - ↳ the Rayleigh criterion thus limits the angular resolution of the telescope.
  - ↳ if a telescope produces images of the order $\frac{1.22\lambda}{D}$, it is diffraction-limited.

Fresnel diffraction

- If we are in the very-near field regime ($R \sim \frac{D^2}{\lambda}$), we can no longer ignore the higher-order phase terms like we did for Fraunhofer diffraction. This is Fresnel diffraction.
- As before, $r \simeq R - \frac{x_0 x + y_0 y}{R} + \frac{x^2 + y^2}{2R}$

  ↳ assume we are on-axis $\therefore x_0 = y_0 = 0$, can change coordinates otherwise.

  ↳ $\frac{x^2+y^2}{2R}$ is no longer negligible



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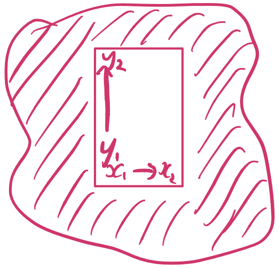
$\therefore \ \Psi_p(0,0) \propto \iint_\Sigma \hat{h}(x,y) \exp\left(ik \frac{x^2+y^2}{2R}\right) dx dy$

↳ this is only tractable for simple apertures.

- Consider a rectangular aperture:

↳ let $u = x\sqrt{\frac{2}{\lambda R}}$ $\quad v = y\sqrt{\frac{2}{\lambda R}}$



$\therefore \ \Psi_p \propto \int_{u_1}^{u_2} \exp\left(\frac{i\pi u^2}{2}\right) du \int_{v_1}^{v_2} \exp\left(\frac{i\pi v^2}{2}\right) dv$

↳ we define the Fresnel Integrals:

$C(w) = \int_0^w \cos\left(\frac{\pi u^2}{2}\right) du \qquad S(w) = \int_0^w \sin\left(\frac{\pi u^2}{2}\right) du$

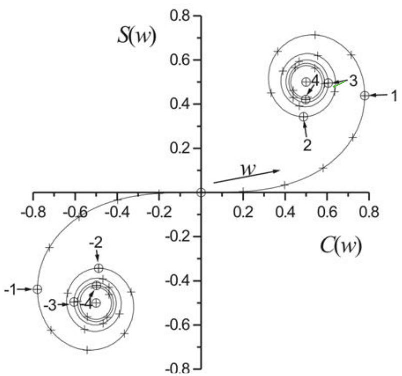
i.e $\boxed{\int_0^w \exp\left(\frac{i\pi u^2}{2}\right) du = C(w) + iS(w)}$

- The locus of $C(w) + iS(w)$ is the Cornu spiral

↳ arc length between points w_1 and w_2 is $w_2 - w_1$, i.e w is the distance from the origin measured along the curve

↳ radius of curvature is $1/\pi w$

↳ curve is odd, and gradually spirals to $\pm(0.5, 0.5)$ as $w \to \infty$



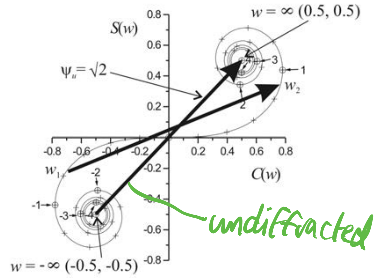
- For a single 1D slit:

$\Psi_p \propto \int_{w_1}^{w_2} \exp\left(\frac{i\pi u^2}{2}\right) du = [C(w_2) + iS(w_2)] - [C(w_1) + iS(w_1)]$

↳ this is equivalent to a vector between points w_1, w_2.

↳ the undiffracted beam is the vector between spiral centres, having length $\sqrt{2}$

↳ intensity $\propto$ square of length



- To find the pattern at other points, the origin must be moved so that it is exactly between S and the observation point, to satisfy the Fresnel conditions

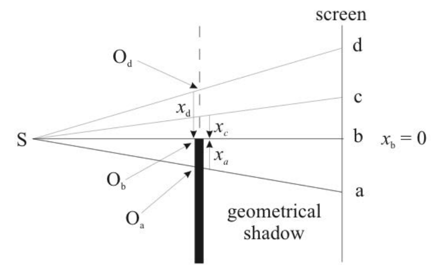
- For diffraction around an edge:

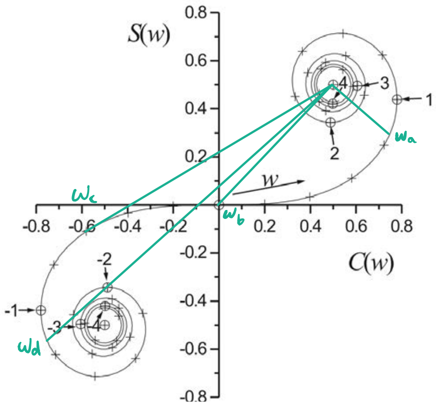
a. $x_2 = \infty \quad x_1 = x_a > 0$

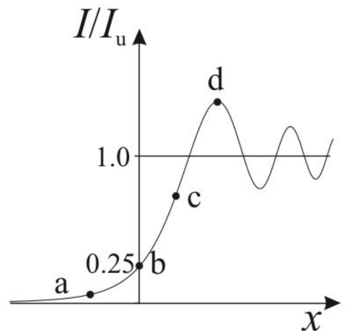
b. $x_2 = \infty \quad x_1 = x_b = 0$

c. $x_2 = \infty \quad x_1 = x_c < 0$

d. $x_3 = \infty \quad x_1 = x_d$







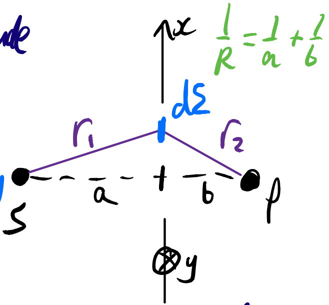
↳ well outside the shadow ($x \to \infty$), intensity $\approx$ undiffracted

↳ amplitude falls as $\sim \frac{1}{w}$ inside the shadow.

- For a narrow finite slit, the integral width is always $\Delta w = d_n\sqrt{2/\lambda R}$ but the starting w_1 changes as the origin moves.
  - ↳ the spanning vector is thus between two points separated by a constant arc length Δw.
- For a wide finite slit, Δw is large so the ends of the spanning vector are in the tightly-spiralled region, hence rapidly oscillating fringes.

Fresnel diffraction for a circular aperture

- The full expression for the diffracted amplitude is found by examining the geometry:



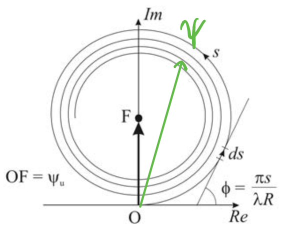
$$\psi_P \propto \iint_\Sigma \frac{h(x,y)K(\theta)\exp\left(ik\frac{x^2+y^2}{2R}\right)}{r_1 r_2}\,dx\,dy$$

↳ making use of circular symmetry, consider the aperture to be composed of annular elements

$s = \rho^2 = x^2+y^2 \qquad dx\,dy = 2\pi\rho\,d\rho = \pi\,ds$

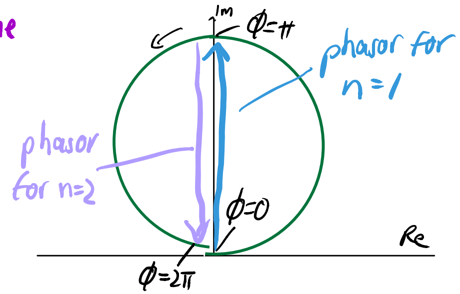
$$\therefore \boxed{\psi_P \propto \int_{s=0}^{s=r_a^2} \frac{K(\theta)\exp\left(\frac{i\pi s}{\lambda R}\right)}{\sqrt{a^2+s}\,\sqrt{b^2+s}}\cdot \pi\,ds}$$

- This integral can be analysed with phasors:
  - ↳ the phase $\phi = \pi s/\lambda R$ increases linearly with s and elemental contributions are of the order ds $\Rightarrow$ approximately circular



  - ↳ $K(\theta)$ decreases with s, since for aperture elements further away from the centre, the point P will be at a greater angle away from undiffracted.
  - ↳ the denominator increases with s, thus the modulus of the integrand decreases with s $\Rightarrow$ radius of the phasor circle is decreasing
  - ↳ the diffracted amplitude is the length of a vector from O to some point a distance s along the curve.
- The diffracted amplitude varies considerably depending on s, from $\psi \approx 0 \rightarrow \psi \approx 2\psi_u$, separated by phase $\phi = \pi$.
  - ↳ the nth Fresnel half-period zone is the annular region between

$$\boxed{\begin{aligned}(n-1)\pi &\le \phi(s) \le n\pi \\ (n-1)\lambda R &\le \rho^2 \le n\lambda R\end{aligned}}$$

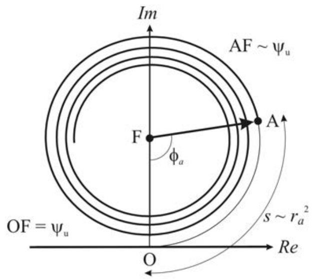


  - ↳ note that odd-numbered zones add to amplitude, while even zones subtract
  - ↳ the area of each zone is the same: $\pi(\rho_n^2-\rho_{n-1}^2) = \pi\lambda R$.
- Neglecting $K(\theta)$ and r_1, r_2 variation (i.e assuming circular phasor diag), each zone contributes equally to the amplitude:
  - ↳ For an aperture of radius r_a, there will be a certain number N of zones, where $r_a^2 = N\lambda R$
  - ↳ if N is even, all zone pairs cancel so $\psi_P \sim 0$.
  - ↳ if N is odd, one zone will remain so $\psi_P \sim 2\psi_u$.

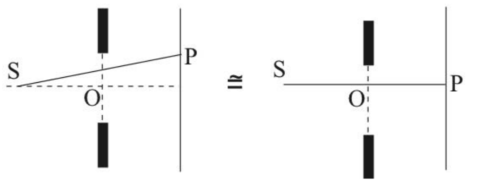
- For large apertures, as the phasor spirals in, the zone contributions decrease and the zones are narrower

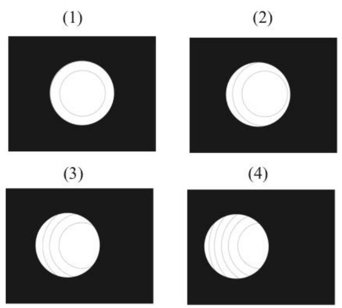
  ✱ Check this!

- Consider a circular obstruction of radius r_a on the axis. The inner zones up to $\rho = r_a$ are obscured, while outer zones are unobstructed.
  - ↳ we thus integrate from $\rho = r_a \rightarrow \rho = \infty$, i.e. from $\phi_a = \frac{\pi r_a^2}{\lambda R} \rightarrow \phi = \infty$
  - ↳ the diffracted amplitude is the length of the vector from A to the centre of the spiral



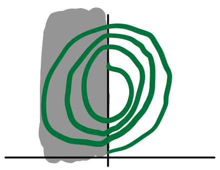
  - ↳ if r_a is not too large (hence ϕ_a not too large) $|AF| \approx |OF|$, hence the diffracted intensity is similar to if there were no obstruction
  - ↳ this is Poisson's spot, a phenomenon that Fraunhofer diffraction (and Babinet's principle) does not explain.
- Off-axis, we can make the approximation that the aperture shifts sideways across the zone structure



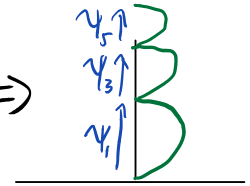


- ↳ there is an oscillation in $|\psi_P|^2$ as P moves off-axis, as the ratio of odd/even zone area changes
- * ↳ the diffraction pattern thus consists of circular fringes with spacing $\approx$ the zone width at the edge of the aperture
- ↳ a long way off-axis, there will be many narrow zones so their contributions cancel → intensity decreases rapidly.

- A Fresnel zone plate blocks alternate half-period zones, resulting in a high intensity
  - ↳ this can be seen by adding half-spirals



$\Rightarrow$



  - ↳ the obstructions should be placed at alternating segments between: $\rho_1 = \sqrt{\lambda R}$, $\rho_2 = \sqrt{2\lambda R}$, $\rho_3 = \sqrt{3\lambda R}$ ...
  - ↳ the net amplitude is $\psi_p \approx 2N\psi_u$ where is the number of open zones in the plate
  - ↳ thus the plate acts as a lens with an effective focal length of $\boxed{f = R = \frac{\rho_n^2}{n\lambda}}$
  - ↳ since $f \propto \frac{1}{\lambda}$, this is a highly chromatic lens
- As point P moves along the axis towards the plate, R decreases. When $R = f/2m$, each open area admits an even number of Fresnel zones, so $\psi_p \rightarrow 0$

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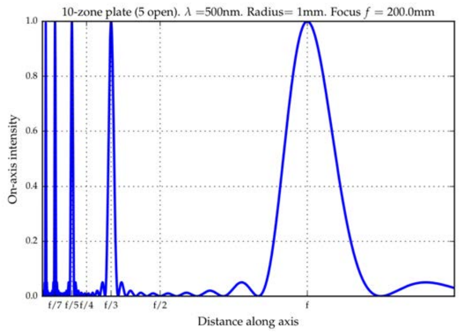
↳ hence there are zeroes at $R = f/2m$

↳ likewise, there are maxim at $R = f/2m+1$, where each zone plate admits an odd number of Fresnel zones so $\Psi_p \to 2N\Psi_n$

↳ in reality, the obliquity factor would reduce the intensity of the maxima



· Although this 'lens' is poor, it may be the only option at high frequencies, since refractive indices → 1.

Interference

· The superposition of two monochromatic waves is:

$$\Psi = Re[\,\psi_1 e^{-i\omega_1 t} + \psi_2 e^{-i\omega_2 t}\,]$$

↳ using $Re[\hat{A}] = \frac{1}{2}(\hat{A} + \hat{A}^*)$, we can expand $I \propto (Re(\hat{\Psi}))^2$ to get:

$$I \propto \frac{1}{2}|\psi_1|^2 + \frac{1}{2}|\psi_2|^2 + Re[\,\psi_1\psi_2^* e^{i(\omega_2-\omega_1)t}\,] + \underbrace{\frac{1}{2}Re[\,\psi_1^2 e^{-2i\omega_1 t} + \psi_2^2 e^{-2i\omega_2 t} + 2\psi_1\psi_2 e^{-i(\omega_1+\omega_2)t}\,]}_{\text{these terms vary more rapidly than the response time of most detectors, hence avg to zero}}$$

↳ the time-average intensity is thus:

$$\langle I \rangle \propto \frac{1}{2}\langle a_1^2\rangle + \frac{1}{2}\langle a_2^2\rangle + \langle a_1 a_2 Re[e^{i[\phi_1-\phi_2-(\omega_1-\omega_2)t]}]\rangle$$

↳ interference phenomena require the third term to be nonzero.

· If the detector averages over a time τ, we will not see interference if $(\omega_1-\omega_2)\tau \gg 1$. i.e we need $\omega_1 \approx \omega_2$

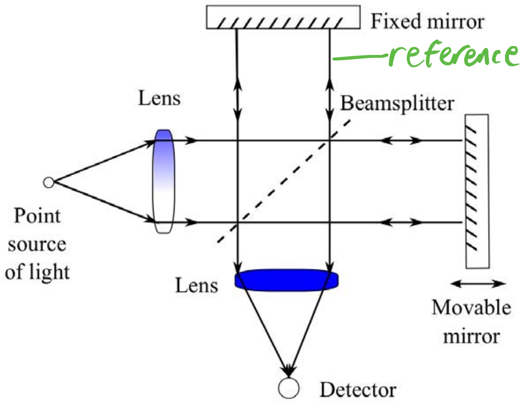
· In practice, ϕ_1 and ϕ_2 of independent sources vary randomly and rapidly – interference is typically only seen when light from a single source is split and recombined, giving a stable $\phi_1 - \phi_2$

· In wavefront division, the interfering waves are derived from different spatial points on a coherent wavefront – e.g slit diffraction.

· In amplitude division, interfering waves are derived by dividing the wavefront's amplitude at a point, e.g reflection/transmission at an interface

The Michelson Interferometer

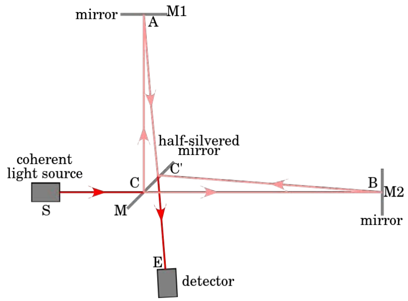
- The Michelson interferometer uses amplitude division:



reference path

↳ the path difference between splitted beams is varied by moving one mirror.
↳ the system must be kept rigid to control the path difference.
↳ there will either be constructive or destructive interference
↳ if we record intensity as a function of mirror position, we see a fringe pattern

- An alternative setup, which also works for extended sources, tilts the mirrors. Hence fringes are seen at the detector, even with both mirrors fixed.



- For a monochromatic point source with $k = 2\pi/\lambda$ $(\omega_1 = \omega_2)$:

$$\langle I \rangle \propto \tfrac{1}{2}\langle a_1^2 \rangle + \tfrac{1}{2}\langle a_2^2 \rangle + \langle a_1 a_2 \, \mathrm{Re}[e^{ikx}] \rangle$$

↳ $Kx = \phi_1 - \phi_2$ is the phase difference, x is the path difference (ie 2x the diff in beamsplitter-mirror distances)

$$I(x) = I_0 (1 + \mathrm{Re}[e^{ikx}])$$ ← averaging implicit

↳ hence the fringe spacing tells us the wavelength.

- If the light is not monochromatic, each wavelength will form its own set of fringes. Total intensity is the sum of fringe pattens.

Fourier transform spectroscopy

- Broadband light (e.g white light) leads to blurred, colourful fringe patterns.
- Let the measured intensity of light in a wavenumber range $k \to k + dk$ be $2S(k)\,dk$. The total intensity at a point is the sum of all waves:

$$I(x) = 2\int_0^\infty S(k)\,(1 + \mathrm{Re}[e^{ikx}])\,dk$$

↳ if we also define $S(k)$ for negative k.

$$I(x) = I_1 + \int_{-\infty}^{\infty} S(k) e^{ikx}\,dk$$

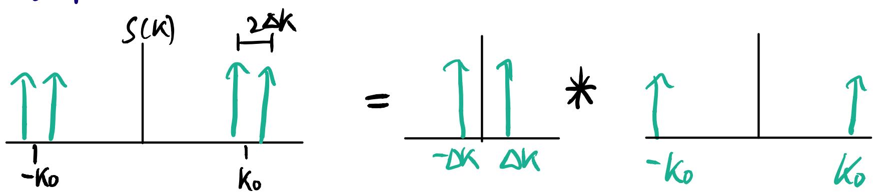
$I_1 = \int_{-\infty}^{\infty} S(k)\,dk$ is the total intensity

↳ thus the spectrum $\propto$ the Fourier transform of intensity

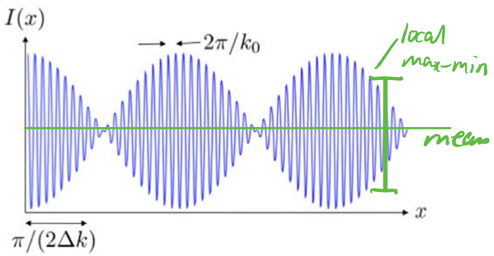
$$\boxed{S(k) \propto \mathcal{F}[I(x) - I_1]}$$

- This result is used in the FT IR spectrometer, which characterises molecules by their vibration frequencies.
- FT spectroscopy is capable of a higher spectral resolution than a diffraction grating, but takes longer since many intensity measurements must be made as a mirror moves.

- If a light source has two closely spaced wavelengths $k_0 \pm \Delta k$, its intensity pattern will be a product of cosines



$\therefore I(x) = I_1(1 + \cos(k_0 x)\cos(\Delta k x))$



- The fringe contrast/visibility quantifies the visibility of the high-freq signal as the ratio of the local min-max disturbance to the mean intensity

$$\text{Visibility} = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}$$

- Δk can be found from the zeroes of the fringe contrast.

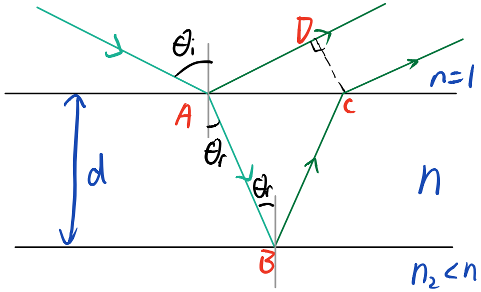
- FT spectroscopy has a finite resolving power because only a finite range of x is sampled:
  - ↳ $I'(x) = I(x) \times W(x)$ ← top hat function, width d
  - ↳ $\therefore S'(k) \propto S(k) * \text{sinc}\left(\frac{xd}{2}\right)$
  - ↳ hence the true spectrum is blurred by a sinc function with width $\Delta k = 2\pi/d$, so the resolving power is:

$$\Rightarrow \boxed{\frac{\lambda}{|\Delta\lambda|} = \frac{k}{\Delta k} = \frac{w}{\lambda}}$$

  - ↳ as with a diffraction grating, resolving power improves as more-distant points on the wavefront are sampled

Thin film interference

- Amplitude division interference can occur naturally when light is incident on a thin film
- The path difference (including the refractive indices):



$$x = n(AB + BC) - AD$$
$$= \frac{2nd}{\cos\theta_r} - 2d\tan\theta_r \sin\theta_i$$

↳ applying Snell's law, $\sin\theta_i = n\sin\theta_r$

$$\Rightarrow \quad x = 2nd\cos\theta_r$$

↳ the phase difference is $kx + \pi$, since A is high→low impedance while B is low→high impedance

- Using the standard interference expression (assuming equal amplitude):

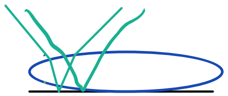
$$\boxed{I(\delta) = I_0\left(1 - \mathrm{Re}[e^{i\delta}]\right), \quad \delta = 2ndk\cos\theta_r}$$

minus from π phase diff

↳ maximum reflection intensity occurs when $\mathrm{Re}[e^{i\delta}] = 0$

$\delta = (2m+1)\pi \Rightarrow nd\cos\theta_r = \frac{(2m+1)}{4}\lambda$

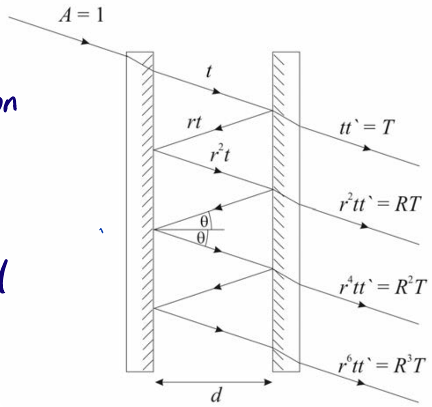
- For an extended source, light will be coming in at many different angles. Thus different reflected angles will correspond to either constructive or destructive interference. These are fringes of equal inclination. If incident beams are near-normal, circular fringes will be observed (Haidinger fringes)
- A more common case is when the films have nonuniform thickness (e.g. soap films). We then observe fringes of equal thickness, i.e. for near-normal incidence bright regions will be seen whenever $2nd = (m+\frac{1}{2})\lambda, \quad m \in \mathbb{Z}$.
  - ↳ another example is if a spherical surface forms an airgap with some other surface. The resulting fringes are Newton's rings



The Fabry-Pelot etalon

- The Fabry-Pelot etalon consists of two half-silvered mirrors sandwiching air. Because the reflection coeff. is high, we must consider interference from multiple beams.

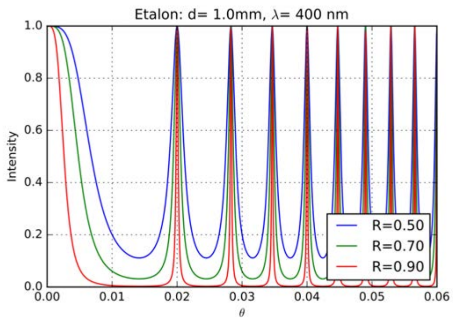
- Assume that both mirrors have reflection coeff r, and transmission coefficients t, t'



  - ↳ each successive beam acquires an amplitude factor $R = r^2$ and a phase shift of $2dk\cos\theta$
  - ↳ the total intensity is the squared sum of the geometric progression:

$$\boxed{|A|^2 = \left|\frac{T}{1 - Re^{i\delta}}\right|^2 = \frac{T^2}{1 + R^2 - 2R\cos\delta}}$$

- The result is a fringe pattern with sharp peaks at $\delta = 2m\pi$, where there is an integer number of half-wavelengths between mirrors.



- We can either use the etalon with normal incidence and vary d, or use an extended source and observe circular fringes.
- At $\delta = 2m\pi$, the max intensity is $\frac{T^2}{1-R^2}$. To find the width of the peaks, it helps to rewrite the intensity as:

$$|A|^2 = \frac{T^2}{(1-R)^2}\left(\frac{1}{1 + \left(4R/(1-R)^2\right)\sin^2(\delta/2)}\right)$$

↳ the width at half-intensity is then given by:

$$\frac{4R}{(1-R)^2}\sin^2(\delta/2) = 1 \Rightarrow \boxed{\delta_{1/2} = \frac{1-R}{\sqrt{R}}}$$ ← small angle approx

↳ the Finesse $\mathcal{F}$ is the ratio of the separation of peaks to their full-width at half maximum $2\delta_{1/2}$

$$\Rightarrow \boxed{\mathcal{F} = \frac{\pi\sqrt{R}}{1-R}}$$

- Hence for high reflection coefficients, the etalon has much better resolution than the Michelson interferometer.

↳ assume that two components can be resolved if they are separated by $2\delta_{1/2}$

$$\delta = 2kd\cos\theta = \frac{4\pi d\cos\theta}{\lambda} \Rightarrow d\delta = -\frac{4\pi d\cos\theta}{\lambda^2}d\lambda$$

at max intensity

$$\therefore \frac{\lambda}{\Delta\lambda} = \frac{2\pi d\cos\theta}{\lambda\,\delta_{1/2}} \Rightarrow \boxed{\frac{\lambda}{\Delta\lambda} = \frac{m\pi}{\delta_{1/2}} = m\mathcal{F}}$$

- An issue in spectroscopy is that neighbouring orders for different wavelengths will overlap - the wavelength diff. at which overlapping occurs is the free spectral range

↳ at normal incidence, peaks are at $2d = m\lambda$

$$\therefore \frac{2d}{\lambda^2}\Delta\lambda \approx \Delta m \Rightarrow \frac{m}{\lambda}(\Delta\lambda)_{fsr} = 1$$ ← set $\Delta m = 1$ by definition of fsr.

$$\Rightarrow \boxed{(\Delta\lambda)_{fsr} = \lambda/m}$$

↳ etalons are ideal for measuring fine structures of narrow spectra.