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Vector Calculus

Suffix notation

- The Cartesian unit vectors form a basis for 3D space. Any vector in the space can uniquely be represented:
  $$\underset{\sim}{a} = a_i \underset{\sim}{e}_i$$
  ↳ orthonormal and right-handed: $\underset{\sim}{e}_1 \cdot (\underset{\sim}{e}_2 \times \underset{\sim}{e}_3) = 1$
- Suffix notation simplifies representations: $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_i$
  ↳ the Kronecker delta δ_{ij} can 'pick out' components, e.g. $\delta_{ij} a_j = a_i$
  ↳ the Levi-Civita symbol ε_{ijk} is $+1$ for even cyclic permutations and -1 for odd. $\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$
  ↳ $\underset{\sim}{a} \times \underset{\sim}{b} = \underset{\sim}{e}_i \varepsilon_{ijk} a_j b_k$
- Matrix quantities with suffix notation:
  ↳ $y = Ax \Rightarrow y_i = A_{ij} x_j$
  ↳ $A = BC \Rightarrow A_{ij} = B_{ik} C_{kj}$
  ↳ $\mathrm{tr} A = A_{ii}$
  ↳ $\det A = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$
- The product of two ε_{ijk} symbols is: $\varepsilon_{ijk}\varepsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix}$
  ↳ most useful contraction:
  $$\boxed{\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}}$$

Vector differential operators in Cartesian coordinates

- The gradient of a scalar field is given by:
  $$\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \underset{\sim}{e}_i \partial_i \phi$$
  shorthand for $\frac{\partial}{\partial x_i}$
  ↳ $\phi(\underset{\sim}{r} + d\underset{\sim}{r}) = \phi(\underset{\sim}{r}) + (\nabla\phi)\cdot d\underset{\sim}{r} + O(|d\underset{\sim}{r}|^2)$
  ↳ the directional derivative at a point on a surface is $\hat{\underset{\sim}{t}} \cdot \nabla\phi$ where $\hat{\underset{\sim}{t}}$ is the unit tangent.
  ↳ so $\nabla\phi$ points in the direction of fastest increase
  ↳ we can construct a normal to the surface $\phi(\underset{\sim}{r}) = \text{const}$ as $\underset{\sim}{n} = \frac{\nabla\phi}{|\nabla\phi|}$
- The divergence of a vector field $\underset{\sim}{F} = \underset{\sim}{e}_i F_i(\underset{\sim}{r})$ is:
  $$\nabla \cdot \underset{\sim}{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} = \partial_i F_i$$
- The curl operator returns another vector field:
  $$\boxed{\nabla \times \underset{\sim}{F} = \underset{\sim}{e}_i \varepsilon_{ijk} \partial_j F_k}$$
  ↳ only defined in 3D space
- The Laplacian can operate on scalar or vector fields:
  $$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_i} \qquad \nabla^2 \underset{\sim}{F} = \underset{\sim}{e}_i \frac{\partial^2 F_i}{\partial x_j \partial x_j}$$
- $\nabla \cdot \underset{\sim}{F}$ and $\nabla^2\phi$ are invariant under rotation of coordinates.

- Important vector calculus identities:

$$\nabla\cdot(\nabla\phi) = \nabla^2\phi$$
$$\nabla\times(\nabla\phi) = \underset{\sim}{0}$$
$$\nabla\cdot(\nabla\times\underset{\sim}{F}) = \underset{\sim}{0}$$
$$\nabla\times(\nabla\times\underset{\sim}{F}) = \nabla(\nabla\cdot\underset{\sim}{F}) - \nabla^2\underset{\sim}{F}$$
$$\nabla\cdot(\underset{\sim}{F}\times\underset{\sim}{G}) = \underset{\sim}{G}\cdot(\nabla\times\underset{\sim}{F}) - \underset{\sim}{F}\cdot(\nabla\times\underset{\sim}{G})$$

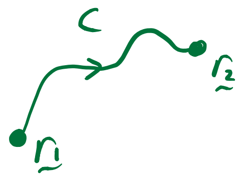
... others can be derived with suffix notation

- $\nabla\times\underset{\sim}{F}=\underset{\sim}{0} \Rightarrow \underset{\sim}{F}=\nabla\phi$ for some ϕ, i.e conservative
- $\nabla\cdot\underset{\sim}{F}=0 \Rightarrow \underset{\sim}{F}=\nabla\times\underset{\sim}{G}$ for some $\underset{\sim}{G}$, i.e solenoidal

Integral theorems

- The gradient theorem:

$$\int_C (\nabla\phi)\cdot d\underset{\sim}{r} = \phi(\underset{\sim}{r}_1)-\phi(\underset{\sim}{r}_2)$$



- The divergence theorem:

$$\int_V (\nabla\cdot\underset{\sim}{F})dV = \int_S \underset{\sim}{F}\cdot d\underset{\sim}{S}$$
← $d\underset{\sim}{S} = \hat{\underset{\sim}{n}}\,dS$ where $\hat{\underset{\sim}{n}}$ points outwards

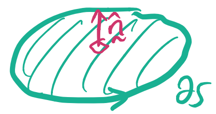
↳ easy to show for cuboid, then build arbitrary surface.

↳ by considering $\nabla\cdot(\phi\underset{\sim}{a})$ and $\nabla\cdot(\underset{\sim}{F}\times\underset{\sim}{a})$ for some constant $\underset{\sim}{a}$, similar results can be derived:

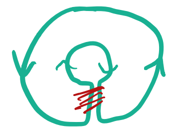
$$\int_V \nabla\phi\, dV = \int_S \phi\, d\underset{\sim}{S} \qquad \int_V(\nabla\times\underset{\sim}{F})dV = \int_S d\underset{\sim}{S}\times\underset{\sim}{F}$$

- Stokes' theorem:

$$\int_S (\nabla\times\underset{\sim}{F})\cdot d\underset{\sim}{S} = \oint_{\partial S} \underset{\sim}{F}\cdot d\underset{\sim}{r}$$



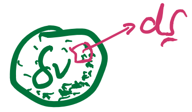
↳ multiply connected surfaces must be treated with care.



- The integral theorems give rise to coordinate-free definitions of div, grad, curl.

$$\hat{\underset{\sim}{t}}\cdot(\nabla\phi) \equiv \lim_{\delta s\to 0}\frac{\delta\phi}{\delta s}$$

$$\nabla\cdot\underset{\sim}{F} = \lim_{\delta V\to 0}\frac{1}{\delta V}\int_{\delta S}\underset{\sim}{F}\cdot d\underset{\sim}{S}$$



$$\hat{\underset{\sim}{n}}\cdot(\nabla\times\underset{\sim}{F}) = \lim_{\delta S\to 0}\oint_{\delta C}\underset{\sim}{F}\cdot d\underset{\sim}{r}$$

Orthogonal curvilinear coordinates

- Cartesian coordinates can be replaced with an independent set:

$$q_1(x_1,x_2,x_3) \qquad q_2(x_1,x_2,x_3) \qquad q_3(x_1,x_2,x_3)$$

- A line element in general coordinates is:

$$d\underset{\sim}{r} = \underset{\sim}{h}_1\, dq_1 + \underset{\sim}{h}_2\, dq_2 + \underset{\sim}{h}_3\, dq_3$$

↳ $\underset{\sim}{h}_i$ can be found by considering how $\underset{\sim}{r}$ changes with an increment in q_i: $\underset{\sim}{h}_i = \dfrac{\partial\underset{\sim}{r}}{\partial q_i}$

↳ we write $\underset{\sim}{h}_i = h_i \underset{\sim}{e}_i$ (no sum), where h_i is the metric coefficient and $\underset{\sim}{e}_i$ is the unit vector.
↳ $h_i = 0$ is a coordinate breakdown.

- The Jacobian matrix describes the $x_i \to q_i$ transform.

$$\begin{bmatrix} \partial x/\partial q_1 & \partial x/\partial q_2 & \partial x/\partial q_3 \\ \partial y/\partial q_1 & \cdots & \\ \partial z/\partial q_1 & & \partial z/\partial q_3 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{h}_1 & \underset{\sim}{h}_2 & \underset{\sim}{h}_3 \end{bmatrix}$$

↳ the Jacobian is the determinant of the matrix:

$$J = \frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)} = \underset{\sim}{h}_1 \cdot (\underset{\sim}{h}_2 \times \underset{\sim}{h}_3)$$

- The volume element in curvilinear coordinates is related to $\underset{\sim}{h}_1 \cdot (\underset{\sim}{h}_2 \times \underset{\sim}{h}_3)$ (volume of paralledipiped) and thus to J

$$dV = |J|\, dq_1 dq_2 dq_3$$

↳ hence Jacobians are needed when changing coordinates in a multiple integral.

- In the case of multiple coordinate transforms, we can just multiply Jacobian matrices. Consider 3 coordinate systems $\alpha_i, \beta_i, \gamma_i$:

$$\frac{\partial \alpha_i}{\partial \gamma_j} = \sum_{k=1}^{n} \frac{\partial \alpha_i}{\partial \beta_k} \frac{\partial \beta_k}{\partial \gamma_j} \Rightarrow J_{\alpha\to\gamma} = J_{\alpha\to\beta} J_{\beta\to\gamma}$$

Orthonormal coordinates

- For orthonormal systems, $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$
- The squared line element is:

$$|d\underset{\sim}{r}|^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$$

↳ there are no cross-terms e.g $dq_1 dq_2$ so it is much easier to do calculus.

- The Jacobian is just $h_1 h_2 h_3$
- Cylindrical coordinates:

$$h_\rho = 1 \qquad h_\phi = \rho \qquad h_z = 1$$

← singular on $\rho = 0$ axis

- Spherical coordinates:

$$h_r = 1 \qquad h_\theta = r \qquad h_\phi = r\sin\theta$$

← singular for $r=0$, $\theta=0$, $\theta=\pi$

- We know that $d\phi = (\nabla\phi)\cdot d\underset{\sim}{r}$. Using the expression for $d\underset{\sim}{r}$ in an orthonormal system, we can show:

$$(\nabla\phi)_i = \frac{1}{h_i}\frac{\partial\phi}{\partial q_i} \quad \text{(no sum)}$$

- Div and curl are more complicated:

$$\nabla\cdot\underset{\sim}{F} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3F_1) + \text{cyclic perms}\right]$$

$$\nabla\times\underset{\sim}{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\underset{\sim}{e}_1 & h_2\underset{\sim}{e}_2 & h_3\underset{\sim}{e}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1F_1 & h_2F_2 & h_3F_3 \end{vmatrix}$$