The Black-Litterman Model

Equilibrium

- Generally, equilibrium means that supply = demand.
- With the Quadratic Utility function and a risk-free asset, the equilibrium portfolio is the CAPM Market portfolio.
- CAPM assumes:
  - every investor agrees on μ and Σ and maximises utility
  - unique risk-free rate of borrowing and lending
  - normally distributed returns

  then $E(r_i) = r_f + \beta_i \left(E(r_m) - r_f \right)$ $\qquad \beta_i = \dfrac{\sigma_{im}}{\sigma_m^2}$
- All investors should thus hold the market portfolio

Prior returns

- BL computes the mkt-implied returns by reverse optimisation
  - ↳ $U = \omega^T \Pi - \frac{\delta}{2} \omega^T \Sigma \omega$
  - ↳ without constraints, this is easy to solve: $\nabla U = 0 \Rightarrow \Pi = \delta \Sigma \omega$
  - ↳ these returns are likely to be 'healthier' than mean historical.
  - ↳ δ can be estimated from the CAPM: $\delta = \dfrac{E(r) - r_f}{\sigma_m^2}$
- The cov matrix of expected returns Σ_Π is modeled by $\tau \Sigma$, where τ is some small scalar (unc in mean << unc in returns)

- The BL prior is then: $E(r) \sim N(\Pi, \tau\Sigma)$, with future returns generated by $r \sim N(E(r), \Sigma)$

Investor's views

- BL allows for K views on N assets, where each view can either be absolute or relative. From these views, we must construct three matrices:
  - ↳ $Q \in \mathbb{R}^{K\times 1}$ is the vector of views
  - ↳ $P \in \mathbb{R}^{K\times N}$ are the asset weights for each view (sum to 0 if relative, 1 otherwise) – a.k.a picking matrix
  - ↳ $\Omega \in \mathbb{R}^{K\times K}$ is the diagonal matrix of view variances
    - → Ω^{-1} is the investor's confidence.
- i.e $PE(r) = Q + \varepsilon$, $\varepsilon \sim N(0, \Omega)$.

Specifying Ω

- He and Litterman (1992) suggest $\Omega = \mathrm{diag}(P(\tau\Sigma)P^T)$, i.e view variance $\propto$ variance of asset returns.
- Alternatively, if a confidence interval is specified, we can extract a variance (assuming normal dist).
- Idzorek's method lets investors specify views with a % confidence

$$\Omega = \alpha P \Sigma P^T, \quad \alpha = \frac{1-\text{conf}}{\text{conf}}$$

The BL-formula from Bayes' Theorem

$$Pr(A|B) = \frac{Pr(B|A)\,Pr(A)}{Pr(B)}$$

sampling dist.

prior dist.

normalising const.

posterior dist.

- In the case of BL:

$$Pr(E(r)\,|\,PE(r)) = \frac{Pr(PE(r)\,|\,E(r))\,Pr(E(r))}{Pr(PE(r))}$$

updated exp. returns

mkt prior

views

↳ but all of these are normal dists, i.e:

$$E(r)\,|\,PE(r) \sim N(\mu^*, M) \quad \} \text{ posterior.}$$

↳ the goal of the BL formula is to compute μ^*

- $E(r) \sim N(\Pi, \tau\Sigma)$ and $PE(r)\,|\,E(r) \sim N(Q, \Omega)$
- Then we can write down the pdfs, e.g

$$f(E(r)) = \frac{1}{\sqrt{(2\pi)^n\,|\tau\Sigma|}} \exp\left[-\frac{1}{2}(E(r)-\pi)^T(\tau\Sigma)^{-1}(E(r)-\pi)\right]$$

- These can be substituted directly into Bayes' formula. Expanding inside the exponent (dropping the $-1/2$)

$$(E(r)-\pi)^T(\tau\Sigma)^{-1}(E(r)-\pi) + (PE(r)-Q)^T\Omega^{-1}(PE(r)-Q)$$

$$= E(r)^T(\tau\Sigma)^{-1}E(r) - \underline{E(r)^T(\tau\Sigma)^{-1}\Pi} - \underline{\Pi^T(\tau\Sigma)^{-1}E(r)} + \Pi^T(\tau\Sigma)^{-1}\Pi$$

$$+ E(r)^T P^T\Omega^{-1}PE(R) - \underline{E(r)^T P^T\Omega^{-1}Q} - \underline{Q^T\Omega^{-1}PE(R)} + Q^T\Omega Q$$

- We can then group equal terms (using symmetry of Ω and $\tau\Sigma$) and factorise $E(r)^T E(r)$ and $E(r)$. We introduce symbols C, H, A:

$$C = (\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q \qquad H = \underbrace{(\tau\Sigma)^{-1} + P^T\Omega^{-1}P}_{\text{symmetrical}}$$

$$A = Q^T\Omega^{-1}Q + \Pi^T(\tau\Sigma)^{-1}\Pi$$

- Then the exponent becomes:

$$E(r)^T H E(r) - 2C' E(r) + A.$$

$$= (HE(r))^T H^{-1} HE(R) - 2C' H^{-1} H E(r) + A$$

$$= (HE(r) - C)^T H^{-1}(HE(r) - C) + A - C^T H^{-1} C$$

$$= (E(r) - H^{-1}C)^T H (E(r) - H^{-1}C) + A - C^T H^{-1} C$$

constant in $E(r)$ so we ignore it

$$\therefore \Pr(E(r) \mid PE(r)) \propto \exp\left[-\frac{1}{2}(E(r) - H^{-1}C)^T H (E(r) - H^{-1}C)\right]$$

$$\therefore E(r) \mid PE(r) \sim N(H^{-1}C, H)$$

$\Rightarrow$

posterior mean: $\mu^* = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}((\tau\Sigma)^{-1}\Pi + P^T\Omega^{-1}Q)$

posterior covariance: $M = ((\tau\Sigma)^{-1} + P^T\Omega^{-1}P)^{-1}$

- However, this covariance is for the expected returns. The posterior estimate for the return dist is $\Sigma^* = \Sigma + M$.

The τ parameter

- τ is measures confidence in the prior estimates
- Can be estimated using confidence intervals: pick a value of τ, compute the 95% or 99% confidence interval and see whether the range of $E(r)$ is reasonable.
- Alternatively, we can set $\tau \sim \frac{1}{T}$, because variance is inversely proportional to the number of samples.