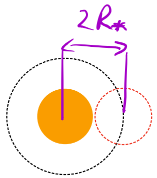
Galactic Dynamics

- Globular clusters are smooth round groups of stars within a galaxy
- We are interested in finding the mass density $\rho(\underline{r})$
  - ↳ measure surface brightness $\mu(\underline{r})$
  - ↳ use M/L relationship to find surface mass density $\Sigma(\underline{r})$
  - ↳ assume spherical symmetry to find $\rho(\underline{r})$
- There are several important radii to describe galaxies:
  - ↳ core radius R_c over which $\rho \sim$ constant.
  - ↳ median radius R_h containing half the light (2D)
  - ↳ tidal radius R_t, for which $\mu \to 0$
- For a collision to occur, there will be one star in the collision volume $\pi(2R_*)^2 v\, t_{coll}$



  - ↳ i.e star density $n_0 = 1/\pi(2R_*)^2 v\, t_{coll}$
  - ↳ collision time is then: $t_{coll} = \dfrac{1}{4\pi R_*^2 v\, n_0}$
  - ↳ sufficiently low that we can assume globular clusters are collisionless.
- Open clusters contain fewer, younger stars and are much smaller than globular clusters.
- Galaxies themselves form clusters.

Orbits

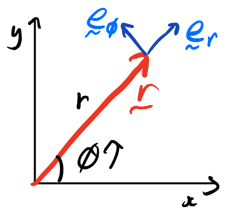
- The goal is to find a self-consistent potential: $\rho(\underline{r})$ implied by the orbits should give rise to ϕ that causes the observed orbits.
  - ↳ if we have many objects, ϕ is approx. smooth
  - ↳ we can average over the orbits and treat both ρ and ϕ as having spatial dependence only.
- The gravitational force per unit mass is $\underline{f} = -\frac{GM}{r^2}\hat{\underline{r}}$
  - ↳ $\underline{f} = -\nabla\phi$, $\phi = -\frac{GM}{|\underline{r}-\underline{r}_1|}$ for a mass at $\underline{r}_1$
- NII: $\underline{F} = m\ddot{\underline{r}} = -m\nabla\phi$
  $\underline{J} = \underline{r} \times (m\dot{\underline{r}}) \Rightarrow \underline{G} = \dot{\underline{J}}$
- The total energy is constant for a given orbit:
  - ↳ $T = \frac{1}{2}m\,\dot{\underline{r}}\cdot\dot{\underline{r}} \Rightarrow \dot{T} = \underline{F}\cdot\dot{\underline{r}} = -m\,\dot{\underline{r}}\cdot\nabla\phi$
  - ↳ but $\frac{d}{dt}\phi(\underline{r}) = \dot{\underline{r}}\cdot\nabla\phi$ by the chain rule

$$\Rightarrow \dot{T} = -m\dot{\phi}$$
$$\Rightarrow \frac{d}{dt}\left(T + m\phi(\underline{r})\right) = 0$$
$$\therefore E = \frac{1}{2}m\,\dot{\underline{r}}\cdot\dot{\underline{r}} + m\phi(\underline{r})$$

- Further, for a central force field, angular momentum is a constant vector so the plane of orbit doesn't change.
  $\dot{\underline{J}} = \underline{r} \times \underline{F} = -m\frac{d\phi}{dr}\,\underline{r}\times\hat{\underline{r}} = \underline{0}$
  - ↳ this reduces orbital problems to 2D.

• Consider the dynamics in plane polars:

↳ For general motion, r and ϕ are changing with ϕ, hence so are $\hat{e}_r$ and $\hat{e}_\phi$.



↳ but the unit vectors can only change orthogonal to themselves

$$\Rightarrow \boxed{\begin{aligned} d\hat{\underline{e}}_r &= r d\phi\, \hat{\underline{e}}_\phi \\ d\hat{\underline{e}}_\phi &= -d\phi\, \hat{\underline{e}}_r \end{aligned}}$$

↳ the velocity can be derived directly by geometry

$$\underline{r} = r\hat{\underline{e}}_r + \phi\,\hat{\underline{e}}_\phi \Rightarrow d\underline{r} = r d\phi\, \hat{\underline{e}}_\phi + dr\,\hat{\underline{e}}_r$$

$$\Rightarrow \boxed{\dot{\underline{r}} = \underbrace{\dot{r}\hat{\underline{e}}_r}_{\text{radial}} + \underbrace{r\dot{\phi}\,\hat{\underline{e}}_\phi}_{\text{tangential}}}$$

• Acceleration in plane polars is given by:

$$\boxed{\ddot{\underline{r}} = (\ddot{r} - r\dot{\phi}^2)\,\hat{\underline{e}}_r + (2\dot{r}\dot{\phi} + r\ddot{\phi})\,\hat{\underline{e}}_\phi}$$

↳ the radial term includes the centrifugal force

↳ the transverse term $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})$ angular momentum per unit mass

• To find the path of the orbit we need to remove t then find $r(\phi)$.

↳ $J/m \equiv h = r^2\dot{\phi}$ and let $u \equiv 1/r$

↳ $\dot{\underline{r}} = -\frac{1}{u^2}\dot{u} = -\frac{1}{u^2}\frac{du}{d\phi}\dot{\phi} = -h\frac{du}{d\phi} \Rightarrow \ddot{r} = -h^2u^2\frac{d^2u}{d\phi^2}$

↳ we can then rewrite the radial equation of motion:

$$f_r = \ddot{r} - r\dot{\phi}^2$$
$$= -h^2\frac{d^2u}{d\phi^2} - \frac{1}{u}h^2u^4$$

$$\Rightarrow \boxed{\frac{d^2u}{d\phi^2} + u = -\frac{f_r}{h^2u^2} = \frac{GM}{h^2}}$$

← the orbit equation in a spherical potential

↳ f_r is a function of u, so we are done.

Kepler orbits

• The solution to the orbit equation is:

$$\frac{l}{r} = 1 + e\cos(\phi - \phi_0)$$

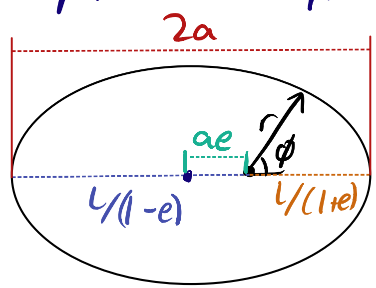
↳ $l = h^2/GM$, e and l are integration constants

• If $e<1$, r is bounded and the path is an ellipse

↳ $\frac{l}{1+e} < r < \frac{l}{1-e}$

↳ $\frac{l}{1+e} + \frac{l}{1-e} = 2a$

$$\Rightarrow l = a(1-e^2)$$



↳ a is the semimajor axis

↳ $\boxed{h^2 = GMa(1-e^2)}$

$e^2 = 1 - \frac{b^2}{a^2}$

↳ the energy per unit mass is $E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\phi}^2 - \frac{GM}{r}$ and can be evaluated anywhere since it is constant

↳ at the periapsis, $\dot{r}=0$, $\dot{\phi} = \frac{h}{r^2}$ $\Rightarrow$ $\boxed{E = \frac{-GM}{2a}}$

- Kepler's laws can be deduced:
  1. Orbits are ellipses with the Sun at a focus
  2. Planets sweep equal areas in equal time:
     $$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\phi} = \frac{h}{2} = \text{const} \quad \text{(for central forces)}$$
  3. $T^2 \propto a^3$: $\frac{\Delta A}{\Delta t} = \frac{h}{2} \Rightarrow T = \frac{2A}{h} = \frac{2\pi a b}{h}$

$$\therefore \quad T = \frac{2\pi a \cdot a\sqrt{1-e^2}}{\sqrt{GMa(1-e^2)}} = 2\pi\sqrt{\frac{a^3}{GM}} \;//$$

Unbound orbits

- If $e \geq 1$, the orbit is unbound (i.e $E \geq 0$)
- The angles ϕ_∞ s.t. $r \to \infty$ are given by
  $\cos\phi_\infty = -1/e$
  ↳ if $e>1$, $\frac{\pi}{2} < \phi_\infty < \pi$ $\therefore$ orbit is a hyperbola
  ↳ if $e=1$, $\phi_\infty = \pm\pi$ $\therefore$ orbit is a parabola.
- As $r \to \infty$, $E \to \frac{1}{2}\dot{r}^2$
  ↳ $L/r = 1 + e\cos\phi$ $\Rightarrow -\frac{L}{r^2}\dot{r} = -e\sin\phi\,\dot{\phi}$
  $\Rightarrow \dot{r} = \frac{eh}{L}\sin\phi$ using $h = r^2\dot{\phi}$
  ↳ so $\boxed{E \to \frac{GM}{2L}(e^2-1)}$ as $r \to \infty$

- If at some point $\underset{\sim}{r_0}$ the particle has velocity $\underset{\sim}{v_0}$ such that $\frac{1}{2}\underset{\sim}{v_0}^2 + \phi(\underset{\sim}{r_0}) > 0$, the particle is able to escape to infinity
  ↳ the escape velocity is then $v_{esc} = \sqrt{-2\phi(\underset{\sim}{r_0})}$

Binary star systems

- With 2 masses, ϕ is no longer fixed at the origin
  $$\phi(\underset{\sim}{r}) = -\frac{GM_1}{|\underset{\sim}{r}-\underset{\sim}{r_1}|} - \frac{GM_2}{|\underset{\sim}{r}-\underset{\sim}{r_2}|}$$
  ↳ let $\underset{\sim}{d} = \underset{\sim}{r_1} - \underset{\sim}{r_2}$
  $$M_1\ddot{\underset{\sim}{r_1}} = -\frac{GM_1M_2}{d^2}\hat{\underset{\sim}{d}}, \quad M_2\ddot{\underset{\sim}{r_2}} = -\frac{GM_1M_2}{d^2}(-\hat{\underset{\sim}{d}})$$
  $$\Rightarrow \ddot{\underset{\sim}{d}} = \ddot{\underset{\sim}{r_1}} - \ddot{\underset{\sim}{r_2}} = -\frac{G(M_1+M_2)}{d^2}\hat{\underset{\sim}{d}}$$
  ↳ equivalent to if we had point mass M_1+M_2 at origin (which we know produces elliptical orbits).
  $$T = 2\pi\sqrt{\frac{a^3}{G(M_1+M_2)}} \quad a \text{ is max separation}$$
- We choose a frame where the COM is stationary
  $\underset{\sim}{r}_{cm} = 0$, $\dot{\underset{\sim}{r}}_{cm} = 0 \Rightarrow \underset{\sim}{r_1} = \frac{M_2}{M_1+M_2}\underset{\sim}{d}$, $\underset{\sim}{r_2} = -\frac{M_1}{M_1+M_2}\underset{\sim}{d}$
  ↳ $\underset{\sim}{J} = \sum_{1,2} M_i \underset{\sim}{r_i} \times \dot{\underset{\sim}{r_i}} = \frac{M_1M_2}{M_1+M_2}\underset{\sim}{d} \times \dot{\underset{\sim}{d}} = \mu\,\underset{\sim}{h}$
  reduced mass

General orbits under radial force

← more general than spherical

$$\frac{d^2u}{d\phi^2}+u=-\frac{f_r}{h^2u^2}, \qquad f_r=-\frac{d\Phi}{dr}=u^2\frac{d\Phi}{du}$$

- There are unbound orbits, with $r\to\infty$ for $\phi\to\phi_\infty$
- For bound orbits, r oscillates between finite limits.

$$\frac{d^2u}{d\phi^2}+u+\frac{1}{h^2}\frac{d\Phi}{du}=0 \qquad \times\ \frac{du}{d\phi}$$

$$\Rightarrow \frac{d}{d\phi}\left[\frac{1}{2}\left(\frac{du}{d\phi}\right)^2+\frac{1}{2}u^2+\frac{\Phi}{h^2}\right]=0$$

$$\therefore\ \frac{1}{2}\left(\frac{du}{d\phi}\right)^2+\frac{1}{2}u^2+\frac{\Phi}{h^2}=\text{const}=\frac{E}{h^2}$$ } dimensional analysis.

$$\boxed{\begin{aligned} E &= \frac{1}{2}h^2\left(\frac{du}{d\phi}\right)^2+\frac{1}{2}h^2u^2+\Phi(r)\\ E &= \frac{1}{2}\dot r^2+\frac{1}{2}r^2\dot\phi^2+\Phi(r)\\ E &= \frac{1}{2}\dot r^2+\frac{h^2}{2r^2}+\Phi(r)\end{aligned}}$$

↳ the apsides are found by $\frac{du}{d\phi}=0$ or $\dot r=0$.

$$E=\frac{1}{2}h^2u^2+\Phi \Rightarrow u^2=\frac{2(E-\Phi)}{h^2}$$ ← quadratic

↳ $u_1=\frac{1}{r_1}$, $u_2=\frac{1}{r_2}$, $r_1<r_2$ WLOG.

↳ r_1 is the pericentre, r_2 is the apocentre

- The radial period T_r is the time for $r_2\to r_1\to r_2$

$$\dot r=\pm\sqrt{2(E-\Phi)-\frac{h^2}{r^2}}$$

$$T_r=\oint dt=2\int_{r_1}^{r_2}\frac{dt}{dr}dr=2\int_{r_1}^{r_2}\frac{dr}{\sqrt{2(E-\Phi)-h^2/r^2}}$$

Precession

- The orbit may have azimuthal motion. This can be found by seeing how ϕ changes during a period

$$\Delta\phi=\oint d\phi=2\int_{r_1}^{r_2}\frac{d\phi}{dt}\frac{dt}{dr}dr=2h\int_{r_1}^{r_2}\frac{dr}{r^2\sqrt{2(E-\Phi)-h^2/r^2}}$$

↳ $\Delta\phi\neq 2\pi$ in one period, so there is a mismatch between the radial period and azimuthal period (time to go round)

↳ define the mean angular velocity as $\bar\omega=\Delta\phi/T_r$ and the mean azimuthal period: $T_\phi=\frac{2\pi}{\bar\omega}=\frac{2\pi}{\Delta\phi}T_r$

↳ if $\frac{\Delta\phi}{2\pi}$ is irrational, the orbit is not closed/periodic.

↳ For a Keplerian orbit, $\Delta\phi=2\pi \Rightarrow T_r=T_\phi$

- In one radial period, the apocentre advances by angle $\Delta\phi-2\pi$ so the 'major axis' rotates at the mean precession rate

$$\boxed{\Omega_p=\frac{\Delta\phi-2\pi}{T_r}}$$

↳ precession is in a sense opposite to the rotation of the star.

↳ no precession for Keplerian orbits

Poisson's Equation

• Relates $\rho(\underline{r})$ to $\Phi(\underline{r})$: $\Phi(\underline{r}) = -\iiint \frac{G\rho(\underline{r}')\,d^3\underline{r}'}{|\underline{r}-\underline{r}'|}$

↳ $\nabla^2\Phi(\underline{r}) = -G\iiint \rho(\underline{r}')\,\nabla^2\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)d^3\underline{r}'$

↳ but $\nabla^2\left(\frac{1}{|\underline{r}-\underline{r}'|}\right) = -4\pi\,\delta(\underline{r}-\underline{r}') \Rightarrow \boxed{\nabla^2\Phi(\underline{r}) = 4\pi G\rho(\underline{r})}$

• If we integrate over any volume V containing a mass M, we get Gauss' Theorem: $\boxed{\int_S \nabla\phi\cdot\hat{\underline{n}}\,dS = 4\pi GM}$

• In spherical systems, $\nabla^2\phi(r) = \frac{1}{r}\frac{d^2}{dr^2}(r\phi)$

↳ finding Φ from ρ is just a matter of solving the differential equation with appropriate B.Cs.

↳ Newton's gravity is linear so we can construct systems by superposition, e.g shell = sphere 1 − sphere 2.

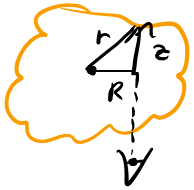
↳ inside a shell, Φ = const. Outside a shell, we have $\Phi = -GM_{shell}/r$ (point particle)

↳ arbitrary spherical density dist. can be analysed by integrating many shells.

$$\Phi(r) = -4\pi G\left[\underbrace{\frac{1}{r}\int_0^r r'^2\rho(r')\,dr'}_{\Phi = -\frac{GM_{enc}}{r}} + \underbrace{\int_r^\infty r'\rho(r')\,dr'}_{\Phi = \frac{G\,dm}{r'} = \text{const.}}\right]$$

Galaxy profiles

• Assume a galaxy has a spherical luminosity density $j(r) = j_0\left(1+\left(\frac{r}{a}\right)^2\right)^{-3/2}$



↳ the surface brightness is the projection of this onto the plane of the sky:

$$I(R) = 2\int_0^\infty j(z)\,dz, \quad r^2 = R^2+z^2$$

$$= 2j_0\int_0^\infty\left[1+\left(\frac{R}{a}\right)^2+\left(\frac{z}{a}\right)^2\right]^{-3/2}dz$$

$$= 2j_0\frac{a^3}{a^2+R^2}\int_0^\infty\frac{dy}{(1+y^2)^{3/2}} \quad \text{with} \quad y = \frac{z}{\sqrt{a^2+R^2}}$$

$$\therefore I(R) = \frac{2j_0 a}{1+R^2/a^2} \leftarrow \text{modified Hubble profile}$$

↳ this is a good fit for elliptical galaxies, so our initial guess for luminosity density is reasonable.

↳ assume density $\propto$ luminosity, $\rho(r) = \rho_0\left(1+\left(\frac{r}{a}\right)^2\right)^{-3/2}$

↳ Φ could be calc'd from Poisson, but this density profile leads to diverging mass.

• A power law density profile $\rho(r) = \rho_0\left(\frac{a}{r}\right)^\alpha$ explains more observations, but also has infinite mass.

• In fact, it is possible to "de-project" the projected density to a spherically-symmetric 3D density

↳ as above, projected from actual density given by:

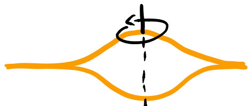
$$I(R) = 2\int_0^\infty j(z)\,dz = 2\int_R^\infty\frac{j(r)\,r\,dr}{\sqrt{r^2-R^2}}$$

↳ can be inverted to give $j'(r) = -\frac{1}{2\pi r}\frac{d}{dr}\int_r^\infty \frac{I(R)\,R\,dR}{\sqrt{R^2-r^2}}$
(an Abel Integral equation)

Nearly circular orbits

- For a circular orbit, $r = R = \text{const}$, $\dot\phi = \Omega = \text{const}$
  ↳ $\ddot r - r\dot\phi^2 = -\frac{d\Phi}{dr} \Rightarrow R\Omega^2 = \frac{d\Phi}{dr}$
  $\Rightarrow \Omega = \sqrt{GM/R^3}$ and $T = 2\pi\sqrt{\frac{R^3}{GM}}$
- For a near-circular orbit, $r = R + \varepsilon(t)$, $\varepsilon \ll R$ and
  $\dot\phi = \Omega + \omega(t)$, $\omega \ll \Omega$
  ↳ $h = R^2\Omega$ must be the same. Expand to first order:
  $R^2\Omega = R^2\Omega + 2R\varepsilon\Omega + R^2\omega \Rightarrow R\omega = -2\varepsilon\Omega$
  ↳ $\ddot r - r\dot\phi^2 = f(r) \Rightarrow \ddot\varepsilon - (R+\varepsilon)(\Omega^2 + 2\Omega\omega) = f(R+\varepsilon)$
  $\Rightarrow \ddot\varepsilon + (3\Omega^2 - f'(R))\,\varepsilon = 0$
  ↳ this is SHM provided $3\Omega^2 - f'(R) > 0$, or equivalently $n < 3$ if $f(R) \propto -R^{-n}$
  ↳ the particle thus has radial oscillation at the epicyclic frequency $\kappa^2 \equiv \Omega_R^2 = 3\Omega^2 - f'(R)$
  ↳ equivalent to an ellipse precessing at rate $\Omega_p = \Omega - \Omega_R$

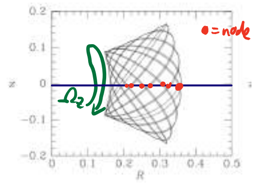
Near-circular orbits in Axisymmetric potentials



- Real density distributions are more often axisymmetric vs spherical.
  ↳ we use cylindrical coordinates: $\rho = \rho(R,z)$, $\Phi = \Phi(R,z)$
  ↳ $E = \left(-\frac{\partial\Phi}{\partial R}, 0, -\frac{\partial\Phi}{\partial z}\right)$

$\Rightarrow \ddot R - R\dot\phi^2 = -\frac{\partial\Phi}{\partial R}$, $R^2\dot\phi = L_z = \text{const}$, $\ddot z = -\partial\Phi/\partial z$
↳ we can remove the $\dot\phi$ term and write the equations in terms of Φ_{eff}, reducing to a 2D problem
$\ddot R = -\partial\Phi_{eff}/\partial R$, $\Phi_{eff} = \Phi + \frac{L_z^2}{2R^2}$
$\ddot z = -\partial\Phi_{eff}/\partial z$

- For a system with plane symmetry $\Phi(R,z) = \Phi(R,-z)$, we can find near-circular orbits: $z = 0$, $R = R_c = \text{const}$, $\dot\phi = \Omega = \text{const}$
  ↳ $\ddot R = 0 \Rightarrow \partial\Phi/\partial R = L_z^2/R^3 \Rightarrow \Omega_c^2 = \frac{1}{R}\,\partial\Phi/\partial R\,\big|_{R=R_c}$
  ↳ for small deviations $R = R_c + x$, $z = z$, $x, z \ll R_c$, we Taylor expand, noting that at $x = z = 0$ we have $\partial\Phi_{eff}/\partial z = 0$ and $\partial\Phi_{eff}/\partial R = 0$
  $$\Phi_{eff}(R_c + x, 0 + z) = \Phi_{eff}(R_c, 0) + \frac{1}{2}x^2\frac{\partial^2\Phi_{eff}}{\partial R^2}\Big|_{(R_c,0)} + \frac{1}{2}z^2\frac{\partial^2\Phi_{eff}}{\partial z^2}\Big|_{(R_c,0)}$$
  ↳ $\ddot R = -\frac{\partial\Phi_{eff}}{\partial R} \Rightarrow \ddot x = -\partial\Phi_{eff}/\partial x = -x\,\partial^2\Phi_{eff}/\partial R^2\big|_{(R_c,0)}$
  ↳ $\ddot z = -\partial\Phi_{eff}/\partial z = -z\,\partial^2\Phi_{eff}/\partial z^2\big|_{(R_c,0)}$
  ↳ we can rewrite this as 2 SHM equations with an epicyclic frequency and vertical frequency:
  $\ddot x = -\kappa^2 x$, $\kappa^2 = \frac{\partial^2\Phi}{\partial R^2}\Big|_{(R_c,0)} + \frac{3L_z^2}{R_c^4}$
  $\ddot z = -\nu^2 z$, $\nu^2 = \frac{\partial^2\Phi}{\partial z^2}\Big|_{(R_c,0)}$
- Hence there is both radial precession at $\Omega_p = \Omega - \kappa$ and nodal precession at $\Omega_z = \Omega - \nu$
  ↳ nodes are the points at which the orbit crosses $z = 0$ upwards



Axisymmetric Potentials

Spherical Coords

- Axisymmetric $\Rightarrow$ $\Phi = \Phi(r, \theta)$
- In the vacuum, $\nabla^2 \Phi = 0$. If we assume a separable form $\Phi = R(r)\,\Theta(\theta)$, we get:

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = 0$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = n(n+1)$$

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{d\Theta}{d\mu}\right] + n(n+1)\Theta = 0, \quad \mu = \cos\theta$$ ← Legendres Equation.

- The radial solution is $R(r) = Ar^n + \frac{B}{r^{n+1}}$, while the angular equation is solved by Legendre polynomials $P_n(x)$
  - $P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}(3x^2 - 1)$
  - $(n+1)P_{n+1}(x) = (2n+1)x P_n(x) - nP_{n-1}(x)$
  - ↳ $P_n(x)$ is oscillatory and forms an orthogonal complete set. $\int_{-1}^{1} P_n(x) P_m(x)\,dx = \frac{2}{2m+1}\delta_{nm}$
  - ↳ can generate with the Rodrigues Formula: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]$
- Outside an axisymmetric body, $R(r) = \frac{B}{r^{n+1}}$. For objects symmetric in $\theta = \pi/2$ ($z=0$), the odd $P_n(x)$ disappear

$$\Phi(r,\theta) = \sum_{k=0}^{\infty} \frac{B_{2k} P_{2k}(\cos\theta)}{r^{2k+1}} = -\frac{GM}{r} + \frac{J_2}{r^3}\frac{1}{2}(3\cos^2\theta - 1) + \frac{J_4}{r^5}P_4(\cos\theta)\ldots$$

↳ for near-spherical bodies, these J terms are perturbations

- We can sometimes find coefficients A_n and B_n by writing an expression for Φ somewhere easy (eg on-axis).
  - ↳ e.g for a ring of matter, $\Phi(z) = -GM/(a^2+z^2)^{1/2}$. We can expand in small z then match terms
  - ↳ gives $\Phi(R,z) \approx -\frac{GM}{a}\left[1 - \frac{1}{4a^2}(2z^2 - R^2) + \ldots\right]$
  - ↳ this potential can be applied to the Earth-Moon system, treating the solar potential as ring-like.

Axisymmetric potentials in cylindrical coordinates

- Consider a thin disk of mass in cylindrical coordinates and seek separable solutions to Poisson's eq: $\Phi(R,z) = J(R)\,Z(z)$
  - ↳ $\frac{d^2 Z}{dz^2} - k^2 Z = 0 \Rightarrow Z(z) = Ae^{kz} + Be^{-kz}$. For finite potentials, $Z(z) = Ae^{-k|z|}$
  - ↳ the radial equation is solved with a Bessell function
    $$\frac{1}{R}\frac{d}{dR}\left(R\frac{dJ}{dR}\right) + k^2 J(R) = 0 \Rightarrow J(R) = J_0(kR),\ Y_0(kR)$$
  - ↳ equivalent of SHM in cylindrical coords.
  - ↳ $\Phi_k = Ce^{-k|z|} J_0(kR)$ ← Y_0 diverges at $R=0$
  - ↳ the general solution is $\Phi(R,z) = \int_0^\infty f(k) e^{-k|z|} J_0(kR)\,dk$ where $f(k)$ is determined by the mass distribution.

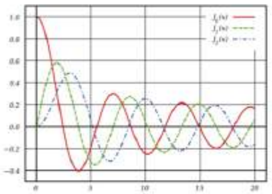
- Bessel functions of the first/second kind, $J_\nu(ks)$ and $Y_\nu(ks)$, solve the ODE:

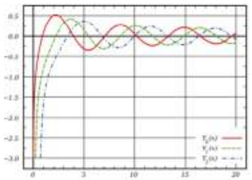
$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) + \left(k^2 - \frac{\nu^2}{s^2}\right) y = 0$$

↳ $J_\nu(0) = 0$ except for $J_0(0) = 1$

↳ $Y_\nu(0) \to -\infty$ for all ν

↳ solution are oscillatory, 'like' sin and cos.



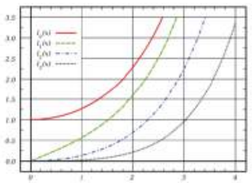


- Modified Bessel functions have $-k^2 y$ instead of $+k^2 y$:

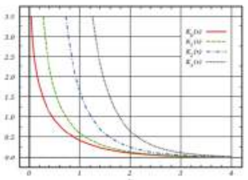
$$\frac{1}{s}\frac{d}{ds}\left(s\frac{dy}{ds}\right) - \left(k^2 - \frac{\nu^2}{s^2}\right) y = 0 \to I_\nu(ks),\ K_\nu(ks)$$

↳ solutions are like cosh and sinh.

I_ν



K_ν



- By analogy to Fourier transforms, we have Hankel transforms with J, Y as the basis:

$$\tilde{g}(k) = \int_0^\infty g(r) J_\nu(kr)\, r\,dr$$

$$g(r) = \int_0^\infty \tilde{g}(k) J_\nu(kr)\, k\,dk \quad \leftarrow \text{inverse}$$

- To find the weighting function, we can construct a Gaussian surface: $\int_V 4\pi G\rho\, dV = \int_V \nabla^2\Phi\, dV = \int_S \nabla\Phi\cdot\hat{n}\, dS$

$$\Rightarrow 4\pi G\Sigma(R) = \left[\frac{\partial\Phi}{\partial z}\right]_{0^-}^{0^+}$$

$$\Rightarrow \Sigma(R) = -\frac{1}{2\pi G}\int_0^\infty f(k) J_0(kR)\, k\,dk$$

$$\Rightarrow f(k) = -2\pi G\int_0^\infty \Sigma(R) J_0(kR)\, R\,dR$$

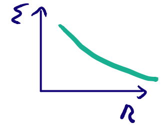
inverse Hankel

- The circular velocity in the plane is $\frac{V_c^2(R)}{R} = \left.\frac{\partial\Phi}{\partial R}\right|_{z=0}$

↳ $\frac{d}{dx}\left[x^m J_m(x)\right] = x^m J_{m-1}(x)$

$$\Rightarrow \frac{V_c^2(R)}{R} = -\int_0^\infty f(k) J_1(kR)\, k\,dk$$

e.g. Mestel Disk: $\Sigma(R) = \frac{\Sigma_0 R_0}{R}$



$$M(<R) = \int_0^R 2\pi\Sigma(R')R'dR' = 2\pi\Sigma_0 R_0 R$$

$$f(k) = -2\pi G\Sigma_0 R_0\int_0^\infty J_0(kR)\,dR$$

$$\Rightarrow \Phi(R,z) = -2\pi G\Sigma_0 R_0\int_0^\infty e^{-k|z|}\frac{J_0(kR)}{k}\,dk$$

$$\frac{V_c^2(R)}{R} = 2\pi G\Sigma_0 R_0\int_0^\infty J_1(kR)\,dk = 2\pi G\Sigma_0 R_0 = \text{const}$$

$$\therefore V_c^2(R) = \frac{GM(R)}{R} \quad \leftarrow \text{same as sphere.}$$

Oort constants

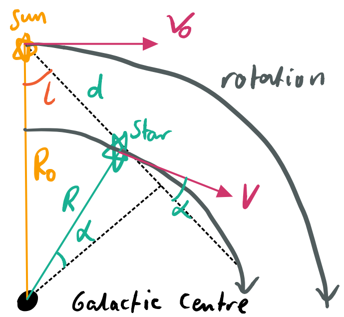
- We model the Milky Way as having stars in circular orbits with $V(R) = R\Omega(R)$
- The radial velocity of the star, seen from the earth, is $V_R = V\cos\alpha - V_0\sin l$

↳ geometry gives $V_R = \left(\frac{V}{R} - \frac{V_0}{R_0}\right) R_0 \sin l$

↳ for nearby stars, $R_0 - R \approx d\cos l$

↳ expanding $\frac{V}{R}$ about R_0, we get $V_R \approx -R_0 \left.\frac{d}{dR}\left(\frac{V}{R}\right)\right|_{R_0} d\sin l\cos l$

$$\Rightarrow \boxed{V_R = Ad\sin 2l}$$



↳ A is the Oort constant $= -\frac{R_0}{2}\frac{d}{dR}\left(\frac{V}{R}\right)\Big|_{R_0} \overset{\text{Quotient}}{=} \frac{1}{2}\left(\frac{V_0}{R_0} - \frac{dv}{dR}\Big|_{R_0}\right)$

↳ A can be determined experimentally by measuring $\frac{V_R}{d}$ as a function of l.

• The tangential velocity (seen from earth) is $V_T = v\sin\alpha - V_0\cos l$

↳ geometry gives $V_T = \left(\frac{V}{R} - \frac{V_0}{R_0}\right)R_0\cos l - \frac{V}{R}d$

$$\simeq -R_0\frac{d}{dR}\left(\frac{V}{R}\right)\Big|_{R_0}\cdot d\cdot\cos^2 l - \frac{V}{R}d$$

↳ define $V_T \approx d(A\cos 2l) + B \Rightarrow B = -\frac{1}{2}\left[\frac{V_0}{R} + \frac{dV}{dR}\Big|_{R_0}\right]$

• Oort constants A, B can be written in terms of Ω

↳ A measures the shear - deviation from 'rigid body'.

$$A = -\frac{R_0}{2}\frac{d}{dR}\left(\frac{V}{R}\right)\Big|_{R_0} \Rightarrow \boxed{A = -\frac{1}{2}R_0\frac{d\Omega}{dR}\Big|_{R_0}}$$

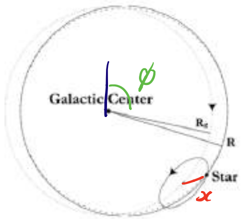
↳ B measures the vorticity - tendency of material to circulate due to differential rotation: $\boxed{B = \left(-\Omega + \frac{1}{2}R_0\frac{d\Omega}{dR}\right)\Big|_{R_0}}$

↳ $\Omega_0 = \frac{V_0}{R_0} = A - B$, $\frac{dv}{dR}\Big|_{R_0} = -(A+B)$

• The epicyclic frequency is given by $K^2 = R\frac{d\Omega^2}{dR^2} + 4\Omega^2$

⇒ $k_0 = \sqrt{-4B(A-B)}$

• We can find the frequencies by comparing the observed velocities with the relative velocities assuming circular motion



$x \equiv R - R_g$, $x(t) = X\cos(\kappa t + \alpha)$

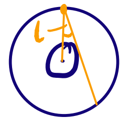
$$\langle(V_\phi - v_c(R_0))^2\rangle / \langle v_r^2\rangle \simeq -\frac{B}{A-B} = \kappa_0^2/4\Omega_0^2$$

The Rotation Curve of our Galaxy

• Neutral H has a 21cm line corresponding to parallel spins becoming antiparallel (low probability)

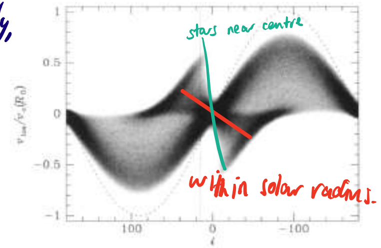
• We can measure Doppler shifts (to get the line-of-sight velocity) for various longitudes l; plotting V_{los} against l gives the rotation curve

• Assuming that $\Omega(R) = \frac{V}{R}$ decreases monotonically, the fastest gas for a given l will be the gas moving in the smallest circle



↳ $R = R_0\sin l$

⇒ $V(R) = V_R(\max) + V_0\sin l$



↳ hence the rotation curve is bounded by a sine curve.

• For spiral galaxies (symmetric potential), the circular velocity $V_c(R)$ is a good measure of the contained mass: $M(<R) \simeq \frac{RV_c^2}{G}$

• Frequently-used spectral lines are:

↳ HI (radio) for neutral gas over large range of radii

↳ $H\alpha$ (optical) for warm gas in inner regions

↳ CO (mm) for the most-inner regions.

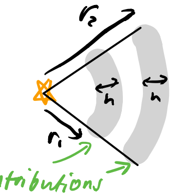
• Difficulties in determining rotation curves:

↳ Beam smearing - points have data from a range of radii so we must deconvolve

↳ Intrinsic: absorption / Finite thickness.

↳ spiral arms are non-axisymmetric

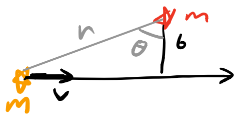
Collisionless Systems



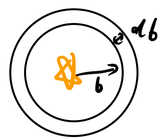
- Gravity is a long range force. Because of the inverse square law, a distant 'shell' of the same thickness has the same force contrib. (Unlike a gas)
- Because of the distance between stars, they almost never physically collide.
- A *collisionless system* is one in which it is a good approx to smooth stars into a mean density $\bar{\rho} \rightarrow \bar{\Phi} \rightarrow$ orbits.

Relaxation time

- The validity of the collisionless approx can be tested by comparing a star's path under a smooth mass dist vs the real path with point stars.
- The *impulse approximation* gives the vertical velocity change as a result of interaction:



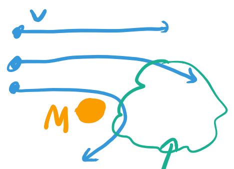
  - ↳ $F_y = \frac{Gm^2 b}{(x^2+b^2)^{3/2}}$, $x = vt$.
  - ↳ $\Delta v_y = \int_{-\infty}^{\infty} F_y(t)\,dt = \frac{Gm}{bv}\int_{-\infty}^{\infty}(1+s^2)^{-3/2}\,ds$
  - ↳ $s = \tan\theta \Rightarrow \Delta v_y = \frac{2Gm}{bv}$.
- The total number of such interactions is the surface density $\times$ the area of band db.



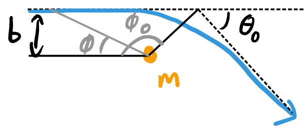
  - ↳ $\delta n = \frac{N}{\pi R^2} \cdot 2\pi b\,db$, where R is the size of the system (eg galaxy), N is num. stars.
  - ↳ by symmetry, the velocity interactions cancel so $\delta v_\perp = 0$, but $\delta v_\perp^2 \neq 0$
- The total change in $v_\perp^2$ is: $\Delta v_\perp^2 = \int_{b_{min}}^{R} 8N\left(\frac{Gm}{Rv}\right)^2 \frac{db}{b}$.
  - ↳ b_{min} is the expected closest approach $\frac{N}{\pi R^2}(\pi b_{min}^2) = 1$ ← 1 per crossing, very approx.
  - ↳ $\boxed{\Delta v_\perp^2 \approx 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda, \quad \Lambda = R/b_{min}}$ after a crossing.
- Collisionless approx valid when $\Delta v_\perp / v \ll 1$; can be shown that this holds for b_{min}.
- The *relaxation time* is the time over which interactions erase memory of the star's initial motion (i.e collisionless approx fails).
  - ↳ $n_{relax}\,\Delta v_\perp^2 \sim v^2$ for memory loss
  - ↳ $v^2 \approx \frac{GNm}{R}$ (circular) $\Rightarrow n_{relax} \sim \frac{N}{8\ln\Lambda} \sim \frac{N}{8\ln N}$
  - ↳ $t_{relax} = n_{relax} \times t_{cross} \approx n_{relax}\frac{R}{v}$
  - ↳ collisionless systems have $t \ll t_{relax}$. True for galaxies but not globular clusters (hence spherical).

Gravitational drag

- The impulse approx. assumes only a vertical impulse, but in reality both $V_\perp$ and $V_\parallel$ change - *dynamical friction*.



- Consider the CoM frame of a large mass M moving at speed v past smaller masses m.
- The deflection can be treated as a Keplerian hyperbolic orbit of m about M.



$$\frac{1}{r} = C\cos(\phi - \phi_0) + \frac{GM}{h^2}$$

↳ the angle ϕ_0 of closest approach can be found by considering $\phi \to 0$ (i.e. $r \to \infty$): $\frac{dr}{dt} \to -v \Rightarrow -v = Cr^2\dot{\phi}\sin(\phi - \phi_0)$

$$\Rightarrow -v = Cbv\sin(-\phi_0)$$

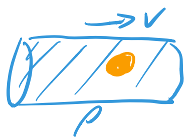
$r \to \infty \Rightarrow 0 = C\cos\phi_0 + \frac{GM}{b^2v^2} \Rightarrow \tan\phi_0 = -bv^2/GM$

↳ the deflection angle $\theta_0 = 2\phi_0 - \pi$

$$\Rightarrow \tan\left(\frac{\theta_0}{2}\right) = \frac{GM}{bv^2}$$

↳ $\theta_0 = \frac{\pi}{2}$ if $b_\perp \sim \frac{GM}{v^2}$

- To estimate drag, assume all stars within cylinder lose their momentum: $M\frac{dv}{dt} = -\pi b_\perp^2 \rho v \cdot v$



$$\Rightarrow \boxed{\frac{dv}{dt} = -\pi\rho\frac{G^2M}{v^2}}$$ ← dynamical friction.

↳ this assumes v much greater than the velocity dispersion of particles in the background.

↳ $F_{fric} \propto M^2$ so wake mass $\propto M$

↳ $F \propto \frac{1}{v^2}$, so drag more relevant for slower bodies.

The Collisionless Boltzmann equation

- Model a system with N particles of mass m (N large) moving under a smooth potential $\Phi(\underline{x}, t)$
- Consider the prob. of finding a star at a particular point in space with a particular velocity – i.e. located in 6D phase space
  - ↳ the full state of the system is specified by the distribution function (i.e. PDF) $f(\underline{x}, \underline{v}, t)$
  - ↳ $\int f(\underline{x}, \underline{v}, t)\, d^3\underline{x}\, d^3\underline{v} = N$ (or can normalise to $=1$).
- Phase space coordinates can be written as $\underline{w} \equiv (\underline{x}, \underline{v}) \equiv (w_1, w_2, \ldots w_6)$
  - ↳ the velocity of phase space flow is $\dot{\underline{w}} = (\dot{\underline{x}}, \dot{\underline{v}}) = (\underline{v}, -\nabla\Phi)$
  - ↳ any flow must conserve the number of stars (or probability)
  - ↳ continuity in $\mathbb{R}^3$: $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{v}) = 0$
  - ↳ continuity in phase space: $\frac{\partial f}{\partial t} + \nabla_{\underline{w}}\cdot(f\dot{\underline{w}}) = 0$
  - ↳ $\nabla_{\underline{w}}\cdot(f\dot{\underline{w}}) = \frac{\partial(f\dot{w}_i)}{\partial w_i} = \dot{w}_i\frac{\partial f}{\partial w_i} + f\frac{\partial \dot{w}_i}{\partial w_i}$
  - ↳ but $\frac{\partial \dot{w}_i}{\partial w_i} = 0$ because $\dot{x}_i = v_i$ indep. of x_i and $\dot{v}_i = \partial\Phi/\partial x_i$ indep. of v_i.

$$\Rightarrow \frac{\partial f}{\partial t} + \sum_{i=1}^{6}\dot{w}_i\frac{\partial f}{\partial w_i} = 0$$ ← can use in other coordinate systems

  - ↳ can separate out $\underline{x}$ and $\underline{v}$ terms to give the Collisionless Boltzmann equation (CBE)

$$\boxed{\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f - \nabla\Phi\cdot\nabla_v f = 0}$$

- Define $\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \dot{w}_i\frac{\partial f}{\partial w_i}$ (with sum), so CBE is $\boxed{\frac{Df}{Dt} = 0}$
  - ↳ this is known as Liouville's theorem
  - ↳ i.e. phase space flow is incompressible (phase-space density conserved)

- Because stars are born and die, phase-space density is not actually conserved: $\frac{Df}{Dt} = B(\underline{x},\underline{v},t) - D(\underline{x},\underline{v},t)$ where B, D are birth/death rates.
  - ↳ acceptable to use CBE when the frac. change in num. stars per crossing time is small: $\gamma = \left| \frac{B-D}{f/t_{cross}} \right| \ll 1$
- In reality, we don't know the distr. function f.
  - ↳ the num. density of stars at location $\underline{x}$ can be found by integrating out velocities: $\nu(x) \equiv \int f(\underline{x},\underline{v})\, d^3\underline{v}$
  - ↳ the pdf of stellar velocities at $\underline{x}$ is $P_x(\underline{v}) = \frac{f(\underline{x},\underline{v})}{\nu(\underline{x})}$
  - ↳ we can only measure $V_{\parallel}$ to our line of sight $\underline{s}$ ($v_{\parallel} = \underline{s}\cdot\underline{v}$), and tangential positions $\underline{x}_{\perp} = \underline{x} - x_{\parallel}\underline{s}$

The Jeans Equations

- Hard to solve the CBE. We can get useful results by finding moments of the CBE (integrating over velocities), giving the Jeans equations
- Zeroth moment returns the continuity equation:

$$\underbrace{\frac{\partial}{\partial t}\int f\, d^3\underline{v}}_{1} + \underbrace{\int v_i \frac{\partial f}{\partial x_i}\, d^3\underline{v}}_{2} - \underbrace{\frac{\partial \Phi}{\partial x_i}\int \frac{\partial f}{\partial v_i}\, d^3\underline{v}}_{3} = 0.$$

1 $= \frac{\partial \nu}{\partial t}$, i.e. time derivative of number density

2 $= \int \frac{\partial}{\partial x_i}(v_i f)\, d^3\underline{v} = \frac{\partial}{\partial x_i}\int v_i f\, d^3\underline{v} = \frac{\partial}{\partial x_i}(\nu \bar{v}_i)$

3 $= \frac{\partial \Phi}{\partial x_i}[f]_{-\infty}^{\infty} = 0$ since $f \to 0$ as $|\underline{v}| \to \infty$

$\Rightarrow$ 1 + 2 + 3 $= \frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}(\nu \bar{v}_i) = 0$ ⟵ Number density conserved.

- First moment gives a fluid equation

$$\underbrace{\frac{\partial}{\partial t}\int f v_j\, d^3\underline{v}}_{1} + \underbrace{\int v_i v_j \frac{\partial f}{\partial x_i}\, d^3\underline{v}}_{2} - \underbrace{\frac{\partial \Phi}{\partial x_i}\int v_j \frac{\partial f}{\partial v_i}\, d^3\underline{v}}_{3} = 0.$$

1 $= \frac{\partial}{\partial t}(\nu \bar{v}_j)$

2 $= \frac{\partial}{\partial x_i}(\nu\, \overline{v_i v_j})$ where $\overline{v_i v_j} = \frac{1}{\nu}\int v_i v_j f\, d^3\underline{v}$

3 $= \cancel{[f v_j]_{-\infty}^{\infty}} - \int \frac{\partial v_j}{\partial v_i} f\, d^3\underline{v} = -\delta_{ij}\nu$

↳ combine terms and subtract $\bar{v}_j \times$ (zeroth order Jeans)

$$\nu\frac{\partial \bar{v}_j}{\partial t} - \bar{v}_j\frac{\partial}{\partial x_i}(\nu \bar{v}_i) + \frac{\partial}{\partial x_i}(\nu\, \overline{v_i v_j}) = -\nu\frac{\partial \Phi}{\partial x_j}$$

↳ define the 'covariance' $\sigma_{ij}^2 = E((v_i - \bar{v}_i)(v_j - \bar{v}_j)) = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$

↳ $$\boxed{\nu\frac{\partial \bar{v}_j}{\partial t} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j} - \frac{\partial}{\partial x_i}(\nu \sigma_{ij}^2)}$$

↳ similar to the fluid equation

$$\rho\frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\nabla)\underline{u} = -\rho\nabla\Phi - \nabla p$$

- σ_{ij}^2 acts like a stress tensor – symmetric so can be diagonalised, with principle components definining a velocity ellipsoid
- The Jeans equations are underdetermined: 9 unknowns (3 for $\bar{\underline{v}}$ and 6 for σ_{ij}^2) but only 4 equations.
- To proceed, make simplifying assumptions:
  - ↳ steady state: $\frac{\partial}{\partial t} = 0$
  - ↳ isotropic: $\sigma_{ij}^2 = \sigma^2(r)\,\delta_{ij}$
  - ↳ non-rotating: $\bar{v}_i = 0$

$$\left.\begin{array}{l}\frac{\partial}{\partial t}=0\\ \sigma_{ij}^2=\sigma^2(r)\delta_{ij}\\ \bar{v}_i=0\end{array}\right\} \Rightarrow \boxed{-\nu\nabla\Phi = \nabla(\nu\sigma^2)}$$

- If we know $\nu(r) \Rightarrow \rho(r) = m\nu(r) \longrightarrow$ find Φ from Poisson $\longrightarrow$ solve for $\sigma^2(r)$ using Jeans
  - ↳ so assuming isotropy, the density dist. gives a consistent model for the velocity structure of the system.

- For axisymmetric systems, use cylindrical polars with $\partial/\partial\phi = 0$. CBE becomes:

$$\frac{\partial f}{\partial t} + v_R \frac{\partial f}{\partial R} + v_z \frac{\partial f}{\partial z} + \left(\frac{v_\phi^2}{r} - \frac{\partial \Phi}{\partial r}\right)\frac{\partial f}{\partial v_R} - \frac{1}{R} v_R v_\phi \frac{\partial f}{\partial v_\phi} - \frac{\partial \Phi}{\partial z}\frac{\partial f}{\partial v_z}$$

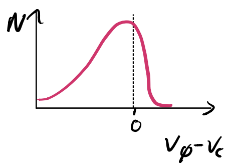
  - ↳ can take moments as before, e.g 0^{th} moment Jeans eq:

$$\frac{\partial \nu}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\nu \bar{v}_R) + \frac{\partial}{\partial z}(\nu \bar{v}_z) = 0$$

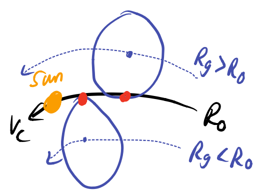
  - ↳ the axisymmetric Jeans eqs explain several galactic phenomena.

Asymmetric drift

- Stars at solar radius tend to lag behind
  - ↳ i.e $V_\phi < V_c$ on average
  - ↳ lag increases with stellar age, suggesting it is a phenomenon that accumulates over time.



- This happens because stars are moving on epicycles with guiding centres at $R = R_g$. Let $\tilde{v}_\phi = v_\phi - v_c$:
  - ↳ stars with $R_g < R_0$ have less angular momentum, so have a lag $\tilde{v}_\phi < 0$



  - ↳ because surface density declines exponentially, there are more stars with $R_g < R_0$, explaining the skew to $\tilde{v}_\phi < 0$
  - ↳ also, velocity dispersion declines with R, so for $R_g < R_0$ there are more epicycles that intersect $R = R_0$.

- Let $V_a \equiv V_c - \bar{V}_\phi$ be the overall asymmetric drift. We can get an expression using the axisymmetric Jeans eq:

$$\frac{\partial(\nu \bar{v}_R)}{\partial t} + \frac{\partial(\nu \overline{v_R^2})}{\partial R} + \frac{\partial(\nu \overline{v_R v_z})}{\partial z} + \nu\left(\frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R}\right) = 0$$

  - ↳ steady state $\Rightarrow \frac{\partial}{\partial t}(\cdot) = 0$
  - ↳ assume planar symmetry and that the sun is on the equatorial plane $\Rightarrow z = 0$, $\partial\nu/\partial z = 0$
  - ↳ define $\sigma_\phi^2 = \overline{v_\phi^2} - (\bar{v}_\phi)^2$ and simplify terms, ignoring V_a^2 terms (small compared to V_c).
  - ↳ result is Stromberg's asymmetric drift equation:

$$V_a \approx \frac{\overline{V_R^2}}{2V_c}\left(\frac{\sigma_\phi^2}{\overline{V_R^2}} - 1 - \frac{\partial \ln(\nu \overline{V_R^2})}{\partial \ln R} - \frac{R}{\overline{V_R^2}}\frac{\partial(\overline{v_r v_z})}{\partial z}\right)$$

  - ↳ all these terms are now observable $\Rightarrow V_a \approx \overline{V_R^2} / (82 \pm 6)\,\mathrm{kms^{-1}}$

- The increasing velocity dispersion over time suggests that something is heating the galactic disk:
  - ↳ MAssive Compact Halo Object (MACHO) originally theorised but no longer considered: would lead to greater heating than observed
  - ↳ most likely a result of galaxy's evolution, i.e increased infall of stars into the galaxy at early times, increasing σ for older stars.

Galactic mass profile

- The mass density in the solar neighbourhood can be estimated from the cylindrical Jeans equation:

$$\frac{1}{R}\frac{\partial(R\nu\overline{v_R v_z})}{\partial R}+\frac{\partial(\nu\overline{v_z^2})}{\partial z}=-\nu\frac{\partial\Phi}{\partial z}$$ steady state so $\frac{\partial}{\partial t}=0$

↳ density falls off much faster vertically, so neglect $\frac{\partial}{\partial R}$ term

$$\Rightarrow \frac{1}{\nu}\frac{\partial}{\partial z}(\nu\overline{v_z^2})=-\frac{\partial\Phi}{\partial z}$$

↳ compare with Poisson's eq, approx'd for thin disk:

$$\frac{\partial^2\Phi}{\partial z^2}=4\pi G\rho \Rightarrow \frac{\partial}{\partial z}\frac{1}{\nu}\frac{\partial}{\partial z}(\nu\overline{v_z^2})=-4\pi G\rho.$$

↳ hence if we had an estimate of ν (does not have to be for all stars - can be e.g G stars) and $\overline{v_z^2}$, we could estimate ρ.

- This technique gives a noisy estimate because we have to differentiate noisy data twice.
  - ↳ instead we can integrate to find $\Sigma(z)$ instead
  $$\Sigma(z)=\int_{-z}^{z}\rho\,dz'=-\frac{1}{2\pi G\nu}\frac{\partial}{\partial z}(\nu\overline{v_z^2})$$
  - ↳ more accurate because only one derivative.
  - ↳ dark matter is needed to explain discrepancies between predicted/observed $\Sigma(z)$.
- For a spherical system (e.g galactic halo), we can derive a Jeans equation: $\frac{1}{\rho_*}\frac{d(\rho_*\sigma_{r,*}^2)}{dr}+\frac{2\beta\sigma_{r,*}^2}{r}=-\frac{d\Phi}{dr}=-\frac{v_c^2}{r}$

↳ β is the velocity anisotropy param. $\beta\equiv1-\frac{\sigma_\theta^2+\sigma_\phi^2}{2\sigma_r^2}=1-\frac{\overline{v_\theta^2}+\overline{v_\phi^2}}{2\overline{v_r^2}}$

↳ given the radial velocity dispersion $\sigma^2_{r,*}$, stellar density ρ_*, and $\beta(r)$, we can uniquely determine the mass profile.

↳ can rewrite as $M(r)=-\frac{r\sigma_r^2}{G}\left[\frac{d\ln\nu}{d\ln r}+\frac{d\ln\sigma_r^2}{d\ln r}+2\beta(r)\right]$

The Virial Theorem

- We can integrate the CBE via $\int(\cdot)x_k\,d^3\underline{x}$ to get a tensor relation:
  - ↳ use the Chandrasekhar PE tensor $W_{jk}\equiv-\int\rho(\underline{x})x_j\frac{\partial\Phi}{\partial x_k}d^3\underline{x}$
  - ↳ the KE tensor is $K_{jk}\equiv\frac{1}{2}\int\rho\overline{v_jv_k}\,d^3\underline{x}$, which is the sum of ordered motion and random motion:
  $$K_{jk}=T_{jk}+\Pi_{jk},\quad T_{jk}\equiv\frac{1}{2}\int\rho\bar{v}_j\bar{v}_k\,d^3\underline{x},\quad \Pi_{jk}\equiv\int\rho\sigma_{jk}^2\,d^3\underline{x}$$
  - ↳ the moment of inertia tensor: $I_{jk}\equiv\int\rho x_jx_k\,d^3\underline{x}$
  - ↳ combine to give the tensor virial theorem
  $$\boxed{\frac{1}{2}\frac{d^2I_{jk}}{dt^2}=2T_{jk}+\Pi_{jk}+W_{jk}}$$
- The tensor virial theorem applies also to self-gravitating collisional systems in the steady state, we can use the scalar virial theorem: $2K+W=0$, $K=\mathrm{trace}(\underline{\underline{T}})+\frac{1}{2}\mathrm{trace}(\underline{\underline{\Pi}})$
- In a stellar system $K=\frac{1}{2}M\langle v^2\rangle \Rightarrow \langle v^2\rangle=\frac{|W|}{M}=\frac{GM}{r_g}$
  - ↳ this gives a simple equation for mass, but sadly $\langle v^2\rangle$, r_g are not readily observable
  - ↳ we only have line of sight velocity dispersion $\langle v_{\parallel}^2\rangle$

Jeans Theorem

- The steady-state CBE is $\underline{v}\cdot\nabla f - \nabla\Phi\cdot\frac{\partial f}{\partial \underline{v}} = 0$, describing continuity in phase space. Orbits are paths in phase space $(x(t), v(t))$
- A constant of motion is a function of $\underline{x}(t), \underline{v}(t), \underline{t}$ that is constant along any orbit: $C(\underline{x}(t_1),\underline{v}(t_1),t_1) = C(\underline{x}(t_2),\underline{v}(t_2),t_2)$
  - ↳ initial condition are constants of motion
  - ↳ e.g $x = ut + x_0$, $C(x,t) = t - \frac{x}{u}$ is a constant of motion
- An integral of motion is a function of phase-space coordinates only that is constant along any orbit

  NO TIME
  - ↳ stronger condition than constant of motion
  - ↳ isolating integrals of motion reduce the dimensionality of the orbit, constraining 6D phase space to a 5D manifold
  - ↳ energy and angular momentum are both isolating.
  - ↳ integrals of motion satisfy $\frac{d}{dt} I(\underline{x}(t),\underline{v}(t)) = 0$

    $$\frac{dI}{dt} = \nabla I\cdot\frac{d\underline{x}}{dt} + \frac{\partial I}{\partial v}\cdot\frac{dv}{dt} = 0$$

    $$\Rightarrow v\cdot\nabla I - \nabla\Phi\cdot\frac{\partial I}{\partial\underline{v}} = 0 \quad \Leftarrow \text{steady-state CBE!}$$

- Jeans' theorem:
  i) Any steady-state solution of the CBE depends on $\underline{x}, \underline{v}$ only through integrals of motion
  ii) Any function of integrals of motion is a solution of the steady-state CBE.

- Proof of Jeans' theorem:
  i) if f is a S-S solution of CBE, $\frac{\partial f}{\partial t} = 0$ by def. So f is an integral of motion $\Rightarrow$ only depends on other integrals
  ii) $\frac{d}{dt}[f(I_1, I_2, \dots I_n)] = \sum_m \frac{\partial f}{\partial I_m}\frac{dI_m}{dt} = 0$

Self-consistent models

- Jeans' theorem: $f(E) = f(\frac{1}{2}v^2 + \Phi(\underline{x}))$ is a solution of the CBE.
  - ↳ assuming all stars have mass m
  - ↳ $\nabla^2\Phi = 4\pi G\rho = 4\pi G m\int f(E)\, d^3\underline{v}$ for a self-consistent model, i.e $f(E)$ is a result of $\Phi(x)$, but $\Phi(x)$ due to $f(E)$.
- Change to relative coordinates to simplify notation:
  - ↳ $\Psi = -\Phi + \Phi_0$, $\mathcal{E} = -E + \Phi_0 = \Psi - \frac{1}{2}v^2$
  - ↳ choose Φ_0 such that $f>0$ for $\mathcal{E}>0$, $f=0$ for $\mathcal{E}\le 0$.
  - ↳ $\nabla^2\Psi = -4\pi G\rho$
- For a spherically-symmetric system, we can get Ψ from $f(\mathcal{E})$:
  - ↳ $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -4\pi G m\int f(\mathcal{E})\, d^3\underline{v} = -4\pi G m\int_0^{\sqrt{2\Psi}} f(\mathcal{E})\, 4\pi v^2\, dv$

    $f \neq 0$ only if $\mathcal{E} = \Psi - \frac{1}{2}v^2 > 0$
  - ↳ $d\mathcal{E} = -v\,dv$ $\therefore$ $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -16\pi^2 G m\int_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi-\mathcal{E})}\, d\mathcal{E}$
- To get $f(\mathcal{E})$ from density,

  $$\nu(\Psi(r)) = \int f\, d^3\underline{v} = 4\pi\int v^2 f(\Psi - \tfrac{1}{2}v^2)\, dv = 4\pi\int_0^{\Psi} f(\mathcal{E})\sqrt{2(\Psi-\mathcal{E})}\, d\mathcal{E}$$

  - ↳ $\frac{d}{d\Psi}$ to give an Abel integral equation with solution

    $$f(\mathcal{E}) = \frac{1}{\sqrt{8}\,\pi^2}\frac{d}{d\mathcal{E}}\int_0^{\mathcal{E}}\left(\frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}}\frac{d\nu}{d\Psi}\right)$$

↳ integration by parts gives *Eddington's formula*:

$$f(\mathcal{E}) = \frac{1}{\sqrt{8}\pi^2}\left[\int_0^{\mathcal{E}} \left(\frac{d\Psi}{\sqrt{\mathcal{E}-\Psi}}\frac{d^2\nu}{d\Psi^2}\right) + \frac{1}{\sqrt{\mathcal{E}}}\left(\frac{d\nu}{d\Psi}\right)_{\Psi=0}\right]$$

Harmonic potential

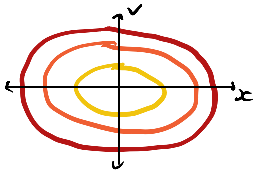
· Inside a constant sphere, the potential is harmonic

$$\Phi = \tfrac{2}{3}\pi G\rho_0(r^2 - 3r_0^2) = \tfrac{1}{2}\omega_0^2(x^2+y^2+z^2) + C$$

· In 1D, this simplifies to:

$$\Phi(x) = \tfrac{1}{2}\omega_0^2 x^2, \quad E = \tfrac{1}{2}v^2 + \tfrac{1}{2}\omega_0^2 x^2 \quad \xRightarrow{\text{Poisson}} \quad \rho(x) = \frac{\omega_0^2}{4\pi G} = \text{const}$$

· Harmonic potentials give ellipses in phase space:



↳ semimajor related to energy

↳ $f(E)$ determines how many phase space orbits of a given amplitude there are

↳ for a self-consistent system, need $f(E)$ to give constant ρ up to x_0 (radius of sphere), $\rho = 0$ outside.

· In relative coordinates, $\Psi = C - \tfrac{1}{2}\omega_0^2 x^2$, $\mathcal{E} = C - \tfrac{1}{2}\omega_0^2 x^2 - \tfrac{1}{2}v^2$

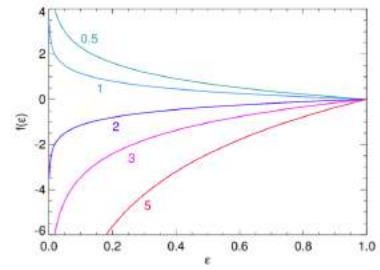
↳ at $x = x_0$, $\mathcal{E} = 0$, $v = 0$ $\Rightarrow$ $C = \tfrac{1}{2}\omega_0^2 x_0^2$

↳ so $\Psi = \tfrac{1}{2}\omega_0^2(x_0^2 - x^2)$, $\mathcal{E} = \Psi - \tfrac{1}{2}v^2$

$$\rho(x) = \int_0^{\sqrt{2\Psi}} f(\mathcal{E})\,dv = \int_0^{\sqrt{\omega_0^2(x_0^2-x^2)}} f(\mathcal{E})\,dv$$

↳ Find f that gives constant ρ by guessing. In this case, $f \sim \frac{1}{\sqrt{\mathcal{E}}}$

Power law distribution functions



$$f = \begin{cases} F\,\mathcal{E}^{n-3/2}, & \mathcal{E} > 0 \\ 0, & \mathcal{E} \le 0 \end{cases}$$

(F const)

· As before, goal is ① get $\rho(\Psi)$, ② $\Psi(r)$ from Poisson ③ $\rho(r)$

① $\rho(r) = \int_0^\infty f(\mathcal{E})\cdot 4\pi v^2 dv = 4\pi F\int_0^{\sqrt{2\Psi}}(\Psi - \tfrac{1}{2}v^2)^{n-3/2} v^2 dv$

↳ can parameterise $v^2 = 2\Psi\cos^2\theta$ so that $\theta \to 0$ gives $v \to \sqrt{2\Psi}$, $\theta \to \frac{\pi}{2}$ gives $v \to 0$.

↳ this gives $\rho(r) = C_n\Psi^n$, $C_n = \dfrac{(2\pi)^{3/2}\,\Gamma(n-\frac{1}{2})\cdot F}{\Gamma(n+1)}$

② Sub into Poisson's equation: $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right) = -4\pi G C_n\Psi$

↳ rescale for convenience:

$$s = r\sqrt{4\pi G C_n \Psi_0^{n-1}}, \quad \psi = \Psi/\Psi_0$$

↳ gives the *Lane-Emden equation* (also used in stars)

$$\boxed{\frac{1}{s^2}\frac{d}{ds}\left(s^2\frac{d\psi}{ds}\right) = \begin{cases} -\psi^n, & \psi > 0 \\ 0, & \psi \le 0 \end{cases}}$$

has analytic solutions for $n = 0, 1, 5$

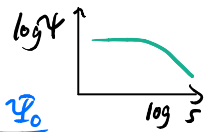
↳ $r = 0$, $\Psi = \Psi_0$ $\Rightarrow$ $s = 0$, $\psi = 1$

↳ $\left.\frac{d\Psi}{dr}\right|_{r=0} = 0$ (no grav. force) $\Rightarrow$ $\left.\frac{d\psi}{ds}\right|_{s=0} = 0$

↳ need $n > 1/2$ to avoid poles of $\Gamma(n - 1/2)$ in C_n

↳ for $n = 5$, this is solved by the *Plummer potential*

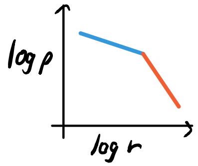
$$\psi = (1 + \tfrac{1}{3}s^2)^{-1/2}$$



③ The Plummer potential results in $\rho = C_5\Psi^5 = \dfrac{C_5\Psi_0}{(1+\frac{1}{3}s^2)^{5/2}}$

- Density in the Plummer potential extends to infinity, but the mass is finite.
  ↳ good model for globular clusters and dwarf spheroidal galaxies
  ↳ not good for elliptical galaxies; drops off too fast.
- To account for dark matter, we may consider two-power law density models, where

$$\rho(r) = \frac{\rho_0}{(r/a)^{\alpha}(1+r/a)^{\beta-\alpha}}, \text{ e.g } \alpha=1, \beta=4$$



Isothermal sphere

- We can model a galaxy as an isothermal sphere, i.e the velocity dispersion $\sigma^2(r)$ is constant.
- We use a Maxwellian distribution function

const

$$f(\mathcal{E}) = \frac{\rho_1}{(2\pi\sigma^2)^{3/2}} \exp\left(\frac{\Psi(r) - \frac{1}{2}v^2}{\sigma^2}\right)$$

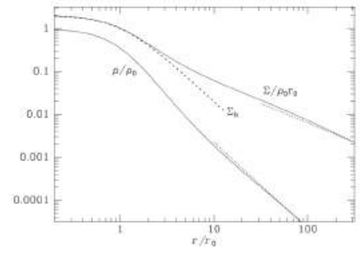
↳ $\rho(r) = \int_0^\infty f(v)\cdot 4\pi v^2 dv = \rho_1 \exp(\Psi/\sigma^2)$

↳ sub into Poisson $\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\ln\rho\right) = -\frac{4\pi G}{\sigma^2}\rho$ using $\Psi(\rho)$

↳ one solution is the singular isothermal sphere $\rho(r) = \frac{\sigma^2}{2\pi G r^2}$

- Singular because $\rho \to \infty$ as $r \to 0$. Also has infinite mass as $r \to \infty$, which is clearly unrealistic.
- Corresponds to a surface density $\Sigma(R) = \frac{\sigma^2}{2GR}$ and a potential $\Phi(r) = 2\sigma^2 \ln r + C$

- We can instead solve Poisson's eq with a B.C $\Psi \to$ const as $r \to 0$ (and $\frac{d\Psi}{dr} = 0$) to avoid a singularity
  ↳ this must be solved numerically.
  ↳ for large r, $\rho \propto r^{-2}$ so the mass still diverges (as does v_{esc})



Star Clusters

- Globular clusters are near-spherical groups of stars as old as their galaxy.
- There are $10^2 - 10^3$ globular clusters in a galaxy

King Models

- The isothermal sphere is a reasonable model at small radii but overestimates density at large radii: weakly bound stars tend to escape.
- We can truncate the Gaussian to give the King models:

$$f(\mathcal{E}) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-3/2}\left(e^{\mathcal{E}/\sigma^2} - 1\right), & \mathcal{E} > 0 \\ 0, & \mathcal{E} \leq 0 \end{cases}$$

- Density and potential found with the usual procedure:

$$\rho(\Psi) = \int_0^{\sqrt{2\Psi}} f(\mathcal{E}) \cdot 4\pi v^2 dv \longrightarrow \frac{d}{dr}\left(r^2 \frac{d\Psi}{dr}\right) = -4\pi G r^2 \rho(\Psi)$$

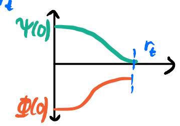
↳ solve numerically. 2 free params: σ^2, $\Psi(r=0)$

↳ as $r\uparrow$ from 0, $\Psi(r)\downarrow$ because $\frac{d^2\Psi}{dr^2} < 0$

↳ as $\Psi \to 0$, the range $[0, \sqrt{2\Psi}]$ shrinks so $\rho \to 0$ at the tidal radius r_t

- There is finite mass within the tidal radius so $\Phi(r_t) = -\frac{GM(r_t)}{r_t}$

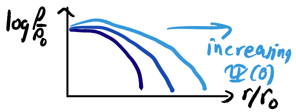
↳ $\Phi(0) = \Phi(r_t) - \Psi(0) \Rightarrow \Psi = -\Phi + \text{const}$



↳ results in family of models parameterised by $\Psi(0)/\sigma^2$

↳ alternatively can use the concentration:

$$c = \log_{10}\left(\frac{r_t}{r_0}\right), \quad r_0 = \sqrt{\frac{9\sigma^2}{4\pi G\rho_0}}$$



Anisotropic velocity distributions

- Thus far we have used energy as an integral of motion.
- To describe systems with anisotropic velocity distributions, we must use angular momentum $L^2 = r^2(v_\theta^2 + v_\phi^2) = r^2 v_\perp^2$:

↳ distribution function $f \equiv f(\mathcal{E}, L^2)$, $v_\theta^2 = v_\phi^2 \neq v_r^2$

↳ we can modify isothermal models using $\mathcal{E} := \mathcal{E} - \frac{L^2}{2r_a^2}$, where r_a is some scale radius.

↳ $$\langle v_\theta^2 \rangle = \langle v_\phi^2 \rangle = \frac{\iiint_{-\infty}^{\infty} v_\theta^2 f(\mathcal{E}, L)\, dv_r dv_\theta dv_\phi}{\iiint_{-\infty}^{\infty} f(\mathcal{E}, L)\, dv_r dv_\theta dv_\phi}$$

$$\Rightarrow \frac{\langle v_\theta^2 \rangle}{\langle v_r^2 \rangle} = \frac{1}{1 + r^2/r_a^2}$$

↳ this model is isotropic at small radii, but anisotropic for $r^2 \gg r_a^2$

- King models can be generalised to give Michie models:

$$f_M(\mathcal{E}, L) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-3/2} \exp\left(-\frac{L^2}{2r_a^2\sigma^2}\right)\left[e^{\mathcal{E}/\sigma^2} - 1\right], & \mathcal{E} > 0 \\ 0, & \mathcal{E} \leq 0 \end{cases}$$

↳ transition from isotropy → anisotropy at $r \simeq r_a$

↳ real clusters show similar behaviour due to collisional effects.

Cluster evolution

- Modelling collisional effects in clusters requires computational methods.
- The Fokker-Planck equation relaxes the CBE to account for changes in phase space density due to interactions:

$\frac{df}{dt} = 0 \rightarrow \frac{df}{dt} = \Gamma(f)$, where $\Gamma(f)$ is the probability of scattering in phase space. ← fast, but hard to find $\Gamma(f)$

- Alternatively, we can directly simulate N-body systems:
  - ↳ can include all kinds of phenomena, e.g. stellar evolution, binaries etc
  - ↳ problem is computational complexity: $O(N^2)$ to calculate forces in each timestep.
  - ↳ make progress with fast computers (GPUs) and numerical approxes.
- For open clusters and the cores of globular clusters, the relaxation time $\ll$ age, so we must consider stellar encounters.

Effects of stellar encounters

1. Relaxation: increase in entropy by energy transfer
   - ↳ transfer from 'hot' → 'cold', where 'hot' means high vel. dispersion
   - ↳ core loses energy to halo, so it must contract. By the virial thm, $M\langle v^2\rangle \approx \frac{GM}{R^2}$ so $R\downarrow \Rightarrow \langle v^2\rangle\uparrow$
   - ↳ core gets 'hotter' as it loses energy ⇒ negative heat capacity
   - ↳ no equilibrium; core continues to get hotter/denser.
2. Stellar escape: cluster evaporation because finite v_{esc}
   - ↳ $v_{esc}^2(\underline{r}) = -2\Phi(\underline{r})$

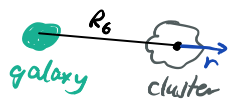
     $$\Rightarrow \langle v_{esc}^2\rangle = \frac{1}{M}\int \rho(\underline{r})\, v_{esc}^2(\underline{r})\, d^3\underline{r} = -\frac{2}{M}\int \rho(\underline{r})\Phi(\underline{r})\, d^3\underline{r}$$

     $= -\frac{4\Omega}{M}$ where Ω is the self-energy ← energy to assemble masses $\rho(\underline{r})$
   - ↳ by the virial thm, $-\Omega = 2T = M\langle v^2\rangle \Rightarrow \boxed{\langle v_{esc}^2\rangle = 4\langle v^2\rangle}$
   - ↳ ε is the fraction of particles with $v_{rms} > v_{esc}$, $\sim 10^{-2}$ for M-B.
   - ↳ evaporation removes $\sim\varepsilon N$ stars on timescale t_{relax}

     $$\frac{dN}{dt} \approx -\frac{\varepsilon N}{t_{relax}} = -\frac{N}{t_{evap}} \Rightarrow t_{evap} = \varepsilon^{-1} t_{relax} \sim 10^2 t_{relax}$$
3. Core collapse
   - ↳ escaping stars 'just' escape, so cluster evolves at constant energy
   - ↳ $E = -kGM^2/R \Rightarrow R \propto M^2 \Rightarrow \rho \propto \frac{M}{R^3} \propto M^{-5}$. Hence as mass is lost, $R \to 0$ and $\rho \to \infty$
   - ↳ because of the negative heat capacity of the core, there is a runaway gravothermal catastrophe (core collapse)
   - ↳ in reality, as $\rho\uparrow$, binaries form → heat source:

     $K_1 + K_2 + K_3 = K_b + E_b + K_3'$, $E_b < 0 \Rightarrow K_b + K_3' > K_1 + K_2 + K_3$
4. Mass segregation: stars have different masses and segregate
   - ↳ stars will have same avg KE so $\langle v^2\rangle \propto m^{-1}$
   - ↳ heavier stars sink to centre; lighter stars → halo.
5. Tidal stripping: cluster stars captured by the galaxy
   - ↳ the tidal force is

     $$F_t = \frac{GM_G}{R_G^2} - \frac{GM_G}{(R_G+r)^2} \approx \frac{2GM_G}{R_G^3} r$$



   - ↳ at the tidal radius, F_t is balanced by attraction to the cluster: $r_t = \left(\frac{M_c}{2M_G}\right)^{1/3} R_G$
6. Binary encounters
   - ↳ For soft (wide) binaries, star #3 is likely travelling faster so transfers energy to the binary: soft binaries get softer, dissolving when $E_b \geq 0$
   - ↳ hard binaries cause strong focussing of #3. An unstable triple forms, eventually ejecting a star. $E_b\downarrow$, so hard binaries get harder.
   - ↳ Heggie's law: soft → softer, hard → harder.
   - ↳ can extract up to $\frac{GM^2}{2R}$ from a binary: only ~100 needed to disrupt cluster.

7. Binary formation : inelastic collisions

↳ dynamical capture results from the interaction of 3 stars in a region $\sim \frac{GM}{\langle v^2 \rangle} \sim 10\,au$ (rare)

↳ tidal capture is when two passing stars creates tides in others' envelopes, dissipating energy. This may result in $E<0 \Rightarrow$ capture

8. Other processes, e.g stellar evolution $\Rightarrow$ mass loss due to stellar winds