Stellar Measurements

- Distances typically measured in parsecs
  ↳ distance at which 1AU subtends one arcsecond
  ↳ can only measure $d \leq 100pc$ by parallax
- Even correcting for the Earth's orbit, we may see that stars move w.r.t distant objects – this is proper motion.
- If we believe two stars have the same absolute mag. then the distances are related by:

$$\underbrace{\frac{F_2}{F_1}}_{\text{received flux}} = 10^{0.4(m_1-m_2)} = \left(\frac{d_1}{d_2}\right)^2$$

- Doppler redshift : $z = \dfrac{\lambda_{obs} - \lambda_0}{\lambda_0}$
  ↳ the radial velocity is then $v = cz$ $(v \ll c)$
- Velocity and position of stars in the galaxy are specified by 6 params: longitude, latitude, distance, radial vel, vel around axis of rotation, vel // to axis

Magnitudes and luminosities

- The effective temp of a star is the temp of a black body whose spectrum most closely matches the star's

$$B_\lambda(T)\ [\underbrace{erg\ s^{-1}}_{\text{power}}\ cm^{-2}\ \underbrace{\mathring{A}^{-1}}_{\text{wavelength spectrum}}\ \underbrace{sr^{-1}}_{\text{solid angle}}] = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT}-1}$$

- The peak of the spectrum is found by $\frac{dB_\lambda}{d\lambda} = 0$

$$\Rightarrow \lambda_{max} T = 0.290\ cm\ K$$

(Wien's displacement law)

↳ as $T\uparrow$, all wavelengths have more power.
↳ the total luminosity is:

$$L = 4\pi R^2 \int B_\lambda(T)d\lambda \int d\Omega$$

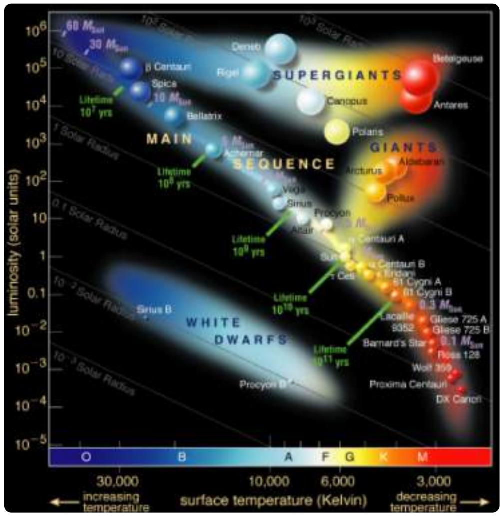
$$= 4\pi R^2 \frac{\sigma T^4}{\pi} \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta d\theta$$

$$\Rightarrow L = 4\pi R^2 \sigma T^4$$

- B-V magnitude is the relative magnitude of B and V filters, which can be used to deduce temp:

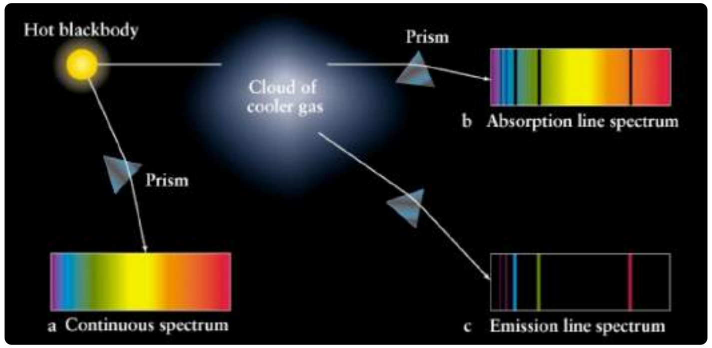
$$B - V = -2.5 \log(F_B/F_V)$$

- The Hertzprung-Russell (HR) diagram plots the absolute magnitude M_V against the B-V mag (equivalently, L against T)

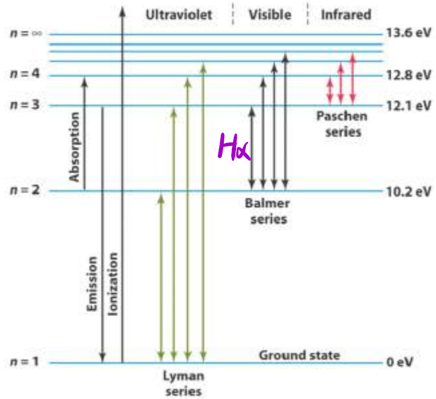


Stellar spectra

· If we observed a black body directly, we would see a continuous spectrum



· Thermal excitation can also produce lines. These are significant when $kT \sim$ ionisation potential



· Certain lines will only be strong if there are enough atoms with the right energy level (e.g. Hα requires $n=2$ electrons).

· Number of atoms in level n given by Boltzmann

$$N_n = A e^{-E_n/kT} g_n$$

← statistical weight $g_n = 2J_n + 1$

↳ total number of atoms is $N = \sum_{n=1}^{\infty} N_n = A\, Z(T)$

↳ relative proportion of ions in consecutive stages of ionisation is given by the Saha equation

vol density of e^- →

$$n_e \frac{N_{i+1}}{N_i} = 2 \frac{Z_{i+1}}{Z_i} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_i/kT}$$

ionisation potential.

↳ can rewrite in terms of the electron pressure $P_e = n_e kT$

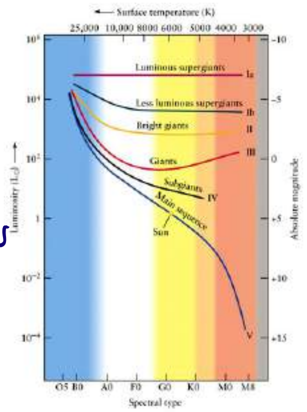
· We can thus use the relative strength of diff lines to gauge the star's temp.

· The Harvard classification is OBAFGKM(LT) with subdivisions from 0 - 9

· The width of an absorption line depends on the density of the stellar atmosphere: less dense ⇒ narrower line

↳ density is related to radius, and thus luminosity (at a given T_{eff})

↳ we can thus use widths to measure the luminosity class, denoted by Roman numerals



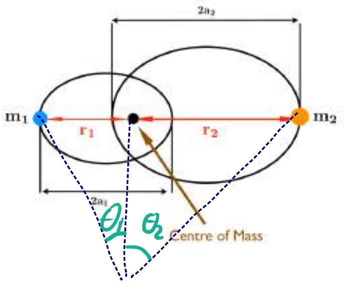
• The luminosities in a stellar cluster can be used to est. the age.

Binary systems and Stellar Mass

• In visual binaries, individual stars can be resolved.

↳ because orbits can be on long timescales, it is difficult to determine if it is truly a binary.

↳ the mass ratio is $\frac{m_1}{m_2} = \frac{r_2}{r_1} = \frac{a_2}{a_1} = \frac{\theta_2}{\theta_1}$

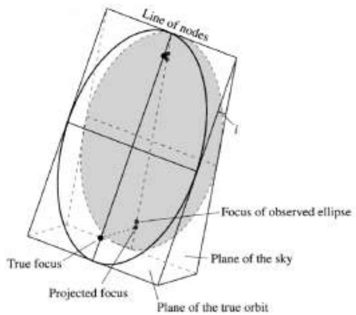


↳ the sum of masses can be found from K3: $P^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)}$

↳ if the orbital incline is at angle i we will observe $\theta' = \theta \cos i$

$$\Rightarrow m_1 + m_2 = \frac{4\pi^2}{G}\left(\frac{d}{\cos i}\right)\frac{\theta'^3}{P^2}$$

↳ to deduce i, we can compare the COM with the apparent focus.



• The majority of known binaries are spectroscopic binaries, whose existence we infer from Doppler-shifted spectral lines:

↳ many binaries have near-circular orbits because of tidal interactions, so orbital velocity is near-constant

$$\frac{m_1}{m_2} = \frac{v_{2r}/\sin i}{v_{1r}/\sin i} = \frac{v_{2r}}{v_{1r}}$$

$$m_1 + m_2 = \frac{P}{2\pi G}\frac{(v_{1r}+v_{2r})^3}{\sin^3 i}$$

↳ we don't know i, so use $\langle \sin^3 i \rangle \approx 0.59$, possibly corrected up to account for selection bias.

• If the second star is very faint (or a black hole/planet), we may only see a single spectrum.

↳ v_{2r} is not observable so we use $v_{2r} = v_{1r} m_1/m_2$

$$\Rightarrow \underbrace{\frac{m_2^3}{(m_1+m_2)^2}\sin^3 i}_{\text{mass function}} = \frac{P}{2\pi G} v_{1r}^3$$

↳ the mass function can put a lower bound on the unseen mass: $m_2 > \frac{P}{2\pi G} v_{1r}^3$

• In an eclipsing binary, there is visual occultation. We can use the light curve to determine the radii

↳ we can deduce the temperature ratio by looking at the drop in flux

- Combining our measurements of M and L for many stars, we see a clearly defined $L \propto M^{3.5}$ relation
  - ↳ this obs must be explained by a theory of stellar structure
  - ↳ stars begin on the H-R main sequence at a location determined by M, then evolve off it.
  - ↳ we can derive the lifetime-mass relation:

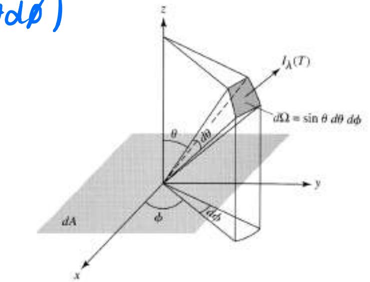
$$\frac{dM}{dt} = kL \quad \therefore \quad t \propto \frac{M}{L} = M^{-2.5}$$

Stellar Atmospheres

- The light we see from a star originates in the photosphere, the layers of gas on the surface. The original source of the energy is gravitational PE.
- The specific intensity is the amount of EM radiation energy with a particular wavelength that passes through a star surface area dA into solid angle $d\Omega$, in time dt:

$$E_\lambda d\lambda = I_\lambda \, d\lambda \, dt (dA \cos\theta)(\sin\theta \, d\theta \, d\phi)$$

erg s^{-1} cm^{-2} $\mathring{A}^{-1}$ sr^{-1}



  - ↳ the mean intensity J_λ is given by integrating over all directions

$$J_\lambda = \frac{1}{4\pi} \int I_\lambda \, d\Omega$$

  - ↳ black bodies radiate isotropically, so $B_\lambda = J_\lambda = I_\lambda$
- The energy density u_λ is $u_\lambda d\lambda = \frac{1}{c} \int I_\lambda \, d\lambda \, d\Omega = \frac{4\pi}{c} J_\lambda \, d\lambda$
  - ↳ for a black body:

$$u_\lambda d\lambda = \frac{8\pi hc/\lambda^5}{e^{hc/\lambda kT} - 1} d\lambda \Rightarrow u = \int_0^\infty u_\lambda \, d\lambda = \frac{4\sigma}{c} T^4$$

- There are many ways to define stellar temp:
  - ↳ effective T_{eff} from luminosity and radius $L = 4\pi R^2 \sigma T_{eff}^4$
  - ↳ excitation T_{ex} from populations of excited states $\frac{n_2}{n_1}$ ← Boltzmann
  - ↳ ionisation T_{ion} from populations of ionisation stages (Saha)

↳ kinetic T_{kin} from the Maxwell-Boltzmann velocity dist.
↳ colour T_{B-V} as the BB temp which best fits observed spectrum
↳ only T_{eff} is a global property (by construction)
↳ in thermodynamic eq, all these Ts are equal.

- In practice, we approx. local thermo. eq (LTE)
  ↳ reasonable when mean free path is small compared to lengths over which pressure and temp change
  ↳ this is true in the stellar interior

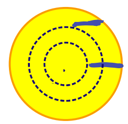
Opacity

- A light beam with intensity I_λ may scatter as it passes through a gas: $dI_\lambda = -\kappa_\lambda \rho I_\lambda ds$ ← distance, gas density
  ↳ κ_λ is the opacity, related to the mean free path μ of the photons $\mu = \frac{1}{\kappa_\lambda \rho} = \frac{1}{n\sigma}$ ← number density × cross section
  ↳ the optical depth is defined as $\tau_\lambda = \int_0^s \kappa_\lambda \rho ds$, i.e the num. of mean free paths from a point to the surface.
  ↳ $I_\lambda = I_{\lambda,0} e^{-\tau_\lambda}$, where $I_{\lambda,0}$ is the intensity in the absence of absorption.
  ↳ gas with $\tau_\lambda >> 1$ is optically thick, else if $\tau_\lambda << 1$ the gas is optically thin
- Sources of opacity:
  1. Bound-bound transitions between electron energy levels (discrete)
  2. Bound-free → photoionisation, when $h\nu > \chi_n$ ← ionisation potential
  3. Free-free → photon absorbed by electron and ion
  4. Thomson scattering → photons scattered by free electrons. Independent of wavelength, but very small cross section, so only impt when high electron density.
  5. H^-, at low temperatures ($T_{eff} \lesssim 7000K$)
- It is helpful to average opacity over all wavelengths, e.g the Rosseland mean opacity $\frac{1}{\langle\kappa\rangle} = \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu(T)}{\partial T} d\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu}$
  ↳ greatest contrib. comes from lowest opacities
  ↳ we can then compare the different sources of opacity

$$\langle \kappa_{bf} \rangle = \kappa_{0,bf}\, \rho\, T^{-3.5}$$
$$\langle \kappa_{ff} \rangle = \kappa_{0,ff}\, \rho\, T^{-3.5}$$
$$\langle \kappa_{es} \rangle = \kappa_{0,es} \frac{1}{\mu_e}$$ ← num electrons per nucleon

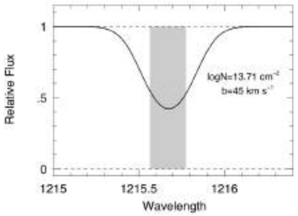
Limb darkening

- Because of the exponential dropoff $I_\lambda = I_{\lambda,0} e^{-\tau_\lambda}$, we only really see photons from depths $\tau_\lambda \simeq 2/3$. This defines the photosphere.
- We therefore see deeper in at the centre, which corresponds to a hotter region



  ↳ thus light from the edges is dimmer and redder
  ↳ this phenomenon is known as limb darkening.

Spectral lines



- The equivalent width W_λ of an absorption line is the width of a rectangle (in units of wavelength) with the same area as the absorption line (height = 100% flux).
  - ↳ $W_\lambda = \int_0^\infty \frac{I_{\lambda,0} - I_\lambda}{I_\lambda} d\lambda = \int_0^\infty (1 - e^{-\tau_\lambda}) d\lambda$
  - ↳ useful because our instruments introduce a broadening convolution, and W_λ is invariant to convolution.
- $\tau_\lambda = \int_0^s n \sigma_\lambda \, ds$ where σ_λ is the interaction cross-section.
  - ↳ σ_λ can be written as a product of an intrinsic cross-section depending on atomic params, and a broadening function (PDF of wavelengths).

$$\sigma_\lambda = \sigma_0 \Phi_\lambda , \quad \sigma_0 = \frac{\lambda^4}{8\pi c} \frac{g_u}{g_l} a_{ul}$$

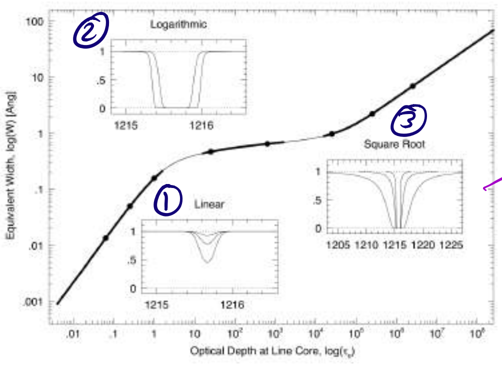
statistical weight of upper energy level

transition prob from lower → upper.

  - ↳ there are several sources of broadening, so absorption lines are never truly 'lines'.
- Natural broadening occurs due to the Heisenberg uncertainty ΔE to the upper energy level: $\Delta E \simeq \frac{\hbar}{\Delta t}$ ← lifetime of level $\propto 1/a_{ul}$
  - ↳ for an atom at rest, $\phi_\lambda = \frac{1}{\pi} \frac{\delta_K}{\delta_K^2 + (\lambda - \lambda_0)^2}$, $\delta_K = \frac{\lambda^2}{4\pi c} \sum_{E_r < E_n} a_{kr}$ ← all transitions
  - ↳ δ_K is the radiation damping constant, inversely proportional to the lifetime of the level k (including sub-jumps).
- Pressure broadening is a result of collisions inducing de-excitation, reducing Δt and increasing ΔE
  - ↳ $\delta_K' = \delta_K + \delta_P$, where $\delta_P = \frac{1}{\Delta t} \approx \frac{v}{\mu} = \sqrt{\frac{2kT}{m}} \cdot n \sigma_c$
  - ↳ because δ_P depends on n, absorption lines in giants (less dense atmospheres) are narrower.
- Doppler broadening is due to a distribution of velocities e.g Maxwell-Boltzmann: $\Psi(v) = \frac{1}{\sqrt{\pi} b} \exp\left[-\frac{(v - v_0)^2}{b^2}\right]$ where $b = \sqrt{2kT/m}$ is the Doppler width due to thermal motion
  - ↳ there may also be bulk motion in the star, leading to Doppler broadening from turbulence
  - ↳ photons may encounter regions of different velocities, causing further broadening from microturbulence
  - ↳ total broadening: $b^2 = b_{th}^2 + b_{turb}^2 + b_{micro}^2$
- The overall broadening is then:

$$\Phi_\lambda = \frac{1}{\pi} \int_0^\infty \frac{\delta_K'}{\delta_K'^2 + [\lambda - \lambda_0(1 + \frac{v}{c})]^2} \Psi(v) \, dv$$

  - ↳ Doppler broadening dominates for $\lambda \simeq \lambda_0$ because $b \gg \delta_K'$, especially for hot stars
  - ↳ But Doppler drops off exponentially, so natural broadening is more impt as λ moves away from λ_0.
  - ↳ alternatively, if N is the column density (i.e num of absorbers in unit cross section $N = \int_0^s n \, ds$), we can write the Voigt function $\boxed{\tau_\lambda = N \sigma_0 \phi_\lambda \otimes \Psi(v)}$
  - ↳ we can then integrate τ_λ to find W_λ, giving the curve of growth



damping wings

① As τ_0 increases from $\tau_\lambda \ll 1$, the line depth increases until all photons are removed from the beam.
↳ line is optically thin
↳ W_λ is a sensitive measure of N

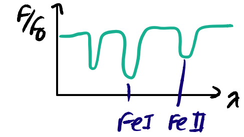
$$W_\lambda = \int_{-\infty}^{\infty} 1 - e^{-\tau_\lambda} d\lambda \approx \int_{-\infty}^{\infty} \tau_\lambda \, d\lambda = N\sigma_0$$

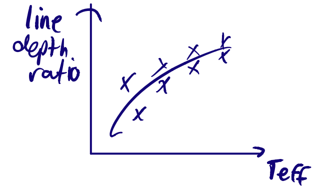
② Logarithmic:
↳ line optically thick
↳ W_λ poor measure of N; sensitive to Doppler param b

③ Square root:
↳ $W_\lambda \propto \sqrt{N}$
↳ damping wings become important.

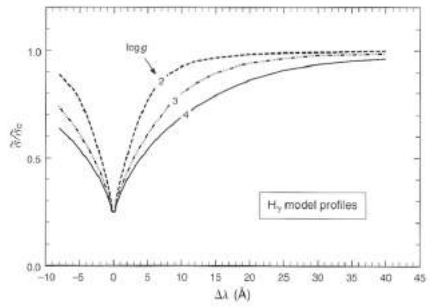
Measuring stellar parameters from spectra

- Temperature can be deduced by looking at metal lines.
- After removing the BB spectrum:
  - ↳ pick two ions with the same expected abundance:
  - ↳ compute line depth ratio
  - ↳ compare this with other stars





- Measuring the wings of lines (affected by pressure broadening) can tell us the surface gravity of a star
  - ↳ Balmer lines are especially sensitive to pressure



- Once temp and surface gravity have been determined, we can find out abundances by comparing the observed spectrum with model spectra of different abundances.

Stellar Radiation

- Newborn stars gain GPE from the collapse of the dust cloud
  - ↳ the virial theorem (for a sys. in equilibrium) states that
    $$-2\langle K\rangle = \langle U\rangle \Rightarrow \langle E\rangle = \langle K\rangle + \langle U\rangle = \tfrac{1}{2}\langle U\rangle$$
  - ↳ i.e 1/2 the δGPE is radiated during contraction, the rest heats the gas.
  - ↳ the GPE to build a star can be found by integrating.
    $$dU = -\frac{G M(r)\,dm}{r} \leftarrow \text{mass of shell}$$
  - ↳ for constant density, $U_g = -\frac{3}{5}\frac{GM^2}{R}$
  - ↳ the virial theorem implies $\Delta E_g = \frac{1}{2}U_g$ can be radiated. But $\Delta E_g / L_\odot$ gives a kelvin-Helmholtz timescale (i.e solar lifetime) 2 orders of mag. too small.
  - ↳ the avg temp of the star can be found using the formula for avg KE per particle $\frac{1}{2} m v_{rms}^2 = \frac{3}{2}k\langle T\rangle$
    $$\Rightarrow \frac{3}{2}k\langle T\rangle \frac{M}{m} = \frac{3}{10}\, GM^2/R$$
    $$\Rightarrow \langle T\rangle = \frac{1}{5}\frac{GMm}{kR} \leftarrow \text{mass of a particle}$$
- The other source of energy is nuclear fusion: there is a mass deficit in the products of fusion (compared to inputs).
  - ↳ the main reaction is $4\,{}^1_1H \rightarrow {}^4_2He + 2e^+ + 2\nu_e$
  - ↳ fusion continues until the Fe-peak; Fe has the highest binding energy per nucleon.
- For nuclei to fuse, they must overcome the Coulomb repulsion until they are close enough for the strong force to dominate.
  - ↳ Classically, temperatures are not high enough for this.
    $$\frac{3}{2}kT = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{r_0}, \quad r_0 \approx 10^{-12}\,m \Rightarrow T \approx 10^{10}\,K.$$
  - ↳ however, in QM, there is positional uncertainty. We can use the de Broglie λ as the min dist for fusion
    $$\frac{3}{2}kT = \frac{1}{4\pi\varepsilon_0}\frac{Z_1 Z_2 e^2}{\lambda}, \quad \lambda = \frac{h}{p}, \quad \frac{p^2}{2\mu m} = \frac{Z_1 Z_2 e^2}{4\pi\varepsilon_0 \lambda}.$$
    $$\Rightarrow T \approx 10^7\,K, \text{ agrees with core of the Sun.}$$

Nuclear reaction rates

- The reaction rate is num reactions / volume / time. Will depend on:
  1. Volume density of reactants
  2. Energy distribution
  3. Prob. of interaction (i.e collision cross-section)

  Just like chem.
- The reaction rate between incoming i and target t is:
  $$\boxed{r_{it} = \int_0^\infty n_i n_t\, \underbrace{\sigma(E)}_{\text{cross-section}}\, v(E)\, \underbrace{\frac{n_E}{n}}_{\text{fraction with energy } E}\, dE}$$
  - ↳ $n_E\,dE$ given by Maxwell-Boltzmann
    $$n_E\,dE = \frac{2n}{\sqrt{\pi}}\frac{1}{(kT)^{3/2}} E^{1/2} \exp\left(-\frac{E}{kT}\right) dE$$

↳ $\sigma(E)$ hard to estimate, but it is an area so has dependence $\sim \lambda^2 \sim \frac{1}{p^2} \sim \frac{1}{E}$. It also depends on the ratio of the Coulomb potential barrier to the KE (for tunnelling. Combined:

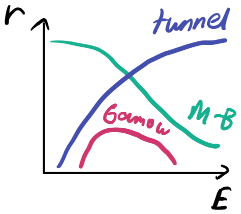
$$\sigma(E) = \frac{S(E)}{E} \exp\left(-\frac{b}{\sqrt{E}}\right)$$

slowly varying function

↳ $v(E) \sim E^{1/2}$

$$\Rightarrow r_{it} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_i n_t}{\sqrt{\mu m \pi}} \int_0^\infty S(E) \exp[-bE^{-1/2}] \exp\left[-\frac{E}{kT}\right] dE$$

↳ there are two competing energy dependencies: tunnelling ↑ at higher E, but M-B says there are fewer particles.



↳ the Gamow peak is at $E_0 = \left(\frac{bkT}{2}\right)^{3/2}$

- There are complications to this model:
  ↳ cross-sections show resonances; some energy transitions are much more likely
  ↳ high densities of free electrons partially shield +ve charge, reducing the Coulomb barrier and ↑ reaction rates.

Nucleosynthesis

- For a nucleosynthetic reaction, we express the power produced as:

$$\varepsilon_{it} = \varepsilon_0 r_0 \underbrace{X_i X_t}_{\text{mass fractions}} \rho^\alpha T^\beta \quad (\text{erg s}^{-1}\text{g}^{-1}), \quad \alpha \approx 1, \ \beta \sim 1 \text{ to } 40.$$

energy/reaction, const, density, temp.

- The most important reaction chain is the pp-chain:

$$4\,{}^1_1H \rightarrow {}^4_2He + 2e^+ + 2\nu_e + 2\gamma$$

↳ 1st step requires a proton to undergo β^+ decay to become a neutron, creating deuterium. This is the slowest step.

$${}^1_1H + {}^1_1H \rightarrow {}^2_1H + e^+ + \nu_e$$
$${}^2_1H + {}^1_1H \rightarrow {}^3_2He + \gamma$$

↳ after this the reaction branches:

69% ppI: ${}^3_2He + {}^3_2He \rightarrow {}^4_2He + 2\,{}^1_1H$

31% ppII: ${}^3_2He + {}^4_2He \rightarrow {}^7_4Be + \gamma \rightarrow {}^7_3Li + \nu_e \xrightarrow{+{}^1_1H} 2\,{}^4_2He$

↳ $\varepsilon_{pp} \propto X^2 \rho T^4$

- The CNO cycle also converts $4\,{}^1_1H \rightarrow {}^4_2He + \dots$ but uses CNO as catalysts:
  ↳ $\varepsilon_{CNO} \propto X \cdot X_{CNO} \rho T^{17}$
  ↳ very strong T-dependence means CNO is dominant when $M \gtrsim 2M_\odot$
  ↳ in stars with lower metallicity, X_{CNO} lower so higher T required for CNO to be the dominant mechanism.
  ↳ each step in CNO proceeds at the same rate (dynamic equilibrium). But $r_{i \to j} \propto n_i \sigma_{i \to j}$, so $r \sim$ const means that $n_i \propto 1/\sigma_{i \to j}$. Nitrogen has the smallest cross section so accumulates.

- As $H \rightarrow He$, the mean molecular weight μ increases. This causes the pressure to decrease:
  ↳ $PV = kT \Rightarrow P = \frac{\rho kT}{\langle m \rangle} = \frac{\rho kT}{\mu m_H}$ ← hydrogen mass in g.

↳ because $P\downarrow$ gravitation causes the core to contract so $\rho\uparrow$ and $T\uparrow$. At a certain point, He nuclei can fuse via the triple alpha reaction:

$$^4He + {}^4He \to {}^8Be + \gamma$$
$$^4He + {}^8Be \to {}^{12}C + \gamma$$

↳ 3α bypasses intermediate elements, explaining the relative abundance of carbon in the universe

↳ $\varepsilon_{3\alpha} \propto Y^3 \rho^2 T^{40}$

- Once there is enough ^{12}C, heavier nuclei form by capturing ^4He

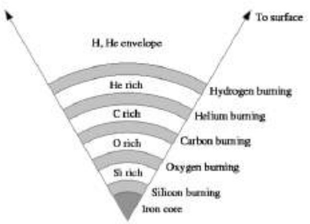
e.g.
$$^{12}_{6}C + {}^{4}_{2}He \to {}^{16}_{8}O + \gamma$$
$$^{16}_{8}O + {}^{4}_{2}He \to {}^{20}_{10}Ne + \gamma$$

↳ the Coulomb barrier is higher for heavier elements, but for $M \geq 8M_\odot$ cores, C and O can burn

- Each step requires higher temp, and the core must contract before the next stage starts.

↳ successive steps also have steeper T-dependence, so occur closer to the centre

↳ result is a stratified (onion-skin) structure.



- Post-Fe elements can be formed by neutron capture (no Coulomb barrier).

↳ depending on neutron flux, we can have the slow s-process or the fast r-process

↳ s-process is repeated absorption/decay, in BAFG stars.

↳ r-process is rapid absorption to form neutron-rich isotopes; dominant in supernovae and neutron star collisions.

Energy Transport in Stars

- Energy generated in the core must find a way to the surface for the star to shine.
- The dominant mechanism depends on the mean-free path μ of photons vs electrons. Because $\mu_\gamma > \mu_e$ in most stars, radiation is more important than conduction

Radiative transport

- Described by the Eddington equation for radiative equilibrium:

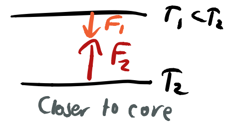
$$\boxed{\frac{dT}{dr} = -\frac{3}{4} \cdot \frac{1}{ac} \cdot \frac{\kappa\rho}{T^3} \cdot \frac{L_r}{4\pi r^2}}$$

↳ a is the radiation constant $= \frac{4\sigma}{c}$ ← Stefan-Boltzmann

↳ κ is opacity

↳ L_r is the luminosity at radius r

- The Eddington eq can be derived by considering LTE for a small cell in the star.



↳ $F_2 \sim \sigma T_2^4$, $F_1 \sim \sigma T_1^4$ $\Rightarrow$ net flux is $F \sim \sigma(T_2^4 - T_1^4)$. Generally, $F \sim -\frac{d}{dr}\sigma T^4$

↳ multiply by photon mean free path $1/\kappa\rho$

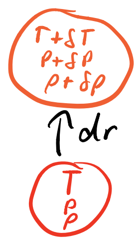
↳ equate flux F to luminosity $-\frac{1}{\kappa\rho}\frac{d}{dr}\sigma T^4 = \frac{L_r}{4\pi r^2}$

↳ additional constant factors come from properly integrating over all angles.

- Near the stellar surface, LTE does not hold so we cannot use the Eddington eq.
- The Eddington equation can approximately relate luminosity and mean temperature:
  - ↳ $L_r = -\frac{4}{3}\frac{1}{\kappa\rho}\cdot 4\pi r^2 ac\, T^3 \frac{dT}{dr}$
  - ↳ integrate and use total luminosity / mean temp
    $L_\odot \approx \frac{1}{3} M\, 4\pi r_\odot\, ac \langle T_\odot \rangle^4$

Convection

- $\kappa \uparrow$ $\frac{dT}{dr}\uparrow$ for constant L_r. But we know κ increases rapidly as temp. decreases, i.e $\langle \kappa \rangle \propto T^{-3.5}$. Hence the temp gradient becomes very steep towards the surface.
- Steep $\frac{dT}{dr}$ is unstable, leading to convection.
- Pressure equilibriates rapidly (because otherwise there is acceleration), but temperature is slow.
  - ↳ equivalent to assuming adiabatic process
  - ↳ convection occurs when the adiabatic process causes a lower temp. gradient vs radiative, giving the Schwarzchild criterion



$$\left|\frac{dT}{dr}\right|_{rad} > \left|\frac{dT}{dr}\right|_{adiabatic} \Rightarrow \text{Convection}$$

i.e if heat flow is slow, convection is the only way to equilibriate.

- From the ideal gas law, $P = K\rho^\gamma$, $\gamma \equiv \frac{C_p}{C_v}$ $(C_p > C_v)$
  $\Rightarrow P\, T^{\gamma/1-\gamma} = \text{const}$

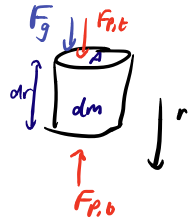
↳ Schwarzchild criterion $\Rightarrow \left|\frac{d\ln P}{d\ln T}\right|_{rad} > \left|\frac{d\ln r}{d\ln P}\right|_{ad}$

$$\Rightarrow \boxed{\left|\frac{d\ln P}{d\ln T}\right|_{star} < \frac{\gamma}{\gamma - 1}}$$

↳ hence when $C_p \simeq C_v$ $(\gamma \simeq 1)$, convection is more likely.

Mixing length theory

- To estimate the convective flux F_c (i.e energy transport due to convection), we need to consider hydrostatic equilibrium and the dynamical timescale.
- In stellar hydrostatic eq., gravity is balanced by pressure:
  - ↳ $A\,dP = -G\frac{M_r \rho A\,dr}{r^2}$ ← ρ locally constant
  - ↳ $\frac{dP}{dr} = -\frac{GM_r\rho}{r^2} \Rightarrow \boxed{\frac{dP}{dr} = -\rho g}$



  - ↳ pressure gradient must be negative (↑ in interior).
  - ↳ the pressure scale height H_P is the radial distance over which pressure drops by a factor of e: $\boxed{\frac{1}{H_P} \equiv -\frac{1}{P}\frac{dP}{dr}} \Rightarrow P = P_0 e^{-r/H_P}$
- The dynamical timescale is the timescale for a star to collapse if there were no pressure.
  - ↳ from SUVAT $t_{ff} = \sqrt{2R/g}$
  - ↳ sub $g = \frac{GM}{R^2}$ with $M = \frac{4}{3}\pi R^3 \langle\rho\rangle$ $\Rightarrow$ $\boxed{t_{dyn} \sim \sqrt{\frac{1}{G\rho}}}$
  - ↳ t_{dyn} can be thought of as the time taken for changes in one part of the star to propagate
  - ↳ alternatively, t_{dyn} is the time to move between equilibrium states.

- During convection, a buoyant hot bubble will rise until it equilibriates. The rising/sinking distance is the mixing length $l = \alpha H_P$, where $\alpha \sim 1$ is a free parameter (we don't know more about α).
- We can model how the heat flow from the bubble changes over a mixing length to arrive at an expression for convective flux

$$F_c = \rho C_P \left(\frac{k}{\mu m_H}\right)^2 \beta^{1/2} \alpha^2 \left(\frac{T}{g}\right)^{3/2} \left[\delta\left(\frac{dT}{dr}\right)\right]^{3/2}$$

↳ in reality, we need to take magnetohydrodynamics into account.

↳ convection leads to mixing between layers as cells can overshoot.

Stellar Models

- Assumptions for static modelling:
  - ↳ spherical symmetry
  - ↳ static (not rotating)
  - ↳ no magnetic field.

  } validity can be argued by saying that departures from ideality are on much longer timescales vs t_{dyn}, so we ignore time.

- A basic stellar model has 4 coupled ODEs:

mechanical {
1. Mass continuity $\frac{dm}{dr} = 4\pi r^2 \rho$
2. Hydrostatic equilibrium $\frac{dP_r}{dr} = -G\frac{M_r \rho}{r^2}$

thermal {
3. Thermal equilibrium $\frac{dL_r}{dr} = 4\pi r^2 \rho \,\varepsilon$
4. Energy transport (choose 1)
   - ↳ Radiative (Eddington) $\frac{dT}{dr} = -\frac{3}{4} \cdot \frac{1}{ac} \frac{\kappa\rho}{T^3} \cdot \frac{L_r}{4\pi r^2}$
   - ↳ Convective $\frac{dT}{dr} = -\frac{\gamma - 1}{\gamma} \frac{\mu m_H}{k} \frac{G M_r}{r^2}$

↳ radial distance r is the independent variable; the above equations are in Euler coordinates

- $\rho, \kappa, \varepsilon$ can be expressed in terms of the fundamental physical characteristics of the plasma (ρ, T, chemistry). This gives the Constitutive relations:

1. Nuclear energy production $\varepsilon \propto \varepsilon_0 \rho^\alpha T^\beta$ ← α, β depend on chemistry
2. Opacity (bound-free) $\langle \kappa_{BF} \rangle = \kappa_{0,BF} T^{-3.5}$
3. Pressure $P = P_g + P_{rad} = \frac{\rho k T}{\mu m_H} + \frac{1}{3} a T^4$

gas pressure ↗ ↖ radiation pressure.

- The gas pressure P_g comes from the ideal gas law:
  - ↳ $P_g V = NkT$, $\rho = N\langle m\rangle/V$, $\langle m\rangle = \mu m_H$
  - ↳ μ is the mean molecular weight, depending on chemical composition. (remember each species contributes different N of p^+, e^-).
  $$\Rightarrow P_g = \rho kT/\mu m_H$$
- Radiation pressure is a result of photon momentum.
  $$P_{rad,\lambda}\, d\lambda = \frac{1}{c}\int_0^{2\pi}\int_0^{\pi} I_\lambda d\lambda \cos^2\theta \sin\theta\, d\theta\, d\phi = \frac{4\pi}{3c} I_\lambda d\lambda$$
  - ↳ for a black-body: $P_{rad} = \frac{4\pi}{3c}\int_0^\infty B_\lambda(T)\, d\lambda = \frac{1}{3} a T^4$
  - ↳ P_{rad} is thus $\frac{1}{3}u$ where u is the energy density. The pressure of an ideal monatomic gas is $\frac{2}{3}u$.

- The equations of stellar structure can instead be formulated in Lagrange coordinates, i.e. in terms of $m = M(r)$
  - ↳ to convert, use the chain rule and $\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$
  - ↳ the equations become:

1.	Mass continuity	$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$
2.	Hydrostatic equilibrium	$\frac{dP}{dm} = -\frac{GM_r}{4\pi r^4}$
3.	Thermal equilibrium	$\frac{dL}{dm} = \varepsilon$
4.	Energy transport	
	↳ Radiative (Eddington)	$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\cdot\frac{L_r}{(4\pi r^2)^2}$
	↳ Convective	$\frac{dT}{dm} = -\frac{\gamma-1}{\gamma}\frac{\mu m_H}{k}\frac{GM_r}{4\pi r^4 \rho}$

Boundary Conditions

- The core of a star has finite density at the core, so at $r=0$, $M_0 = 0$ and $L_r = 0$.
- We can approximate stars as having a clear-cut surface at $r = R_*$, so $P(R_*) = 0$ and $T(R_*) = 0$.
  - ↳ this is justified because $P(R_*) \ll \langle P\rangle$ and likewise for T
  - ↳ may be more appropriate to use $T(R_*) = \left(L/4\pi\sigma R_*^2\right)^{1/4}$ ← effective temp
- The 4 equations cannot be solved analytically without making major assumptions, e.g. an adiabatic pressure $P = K\rho^\gamma$ leads to a family of solutions called polytropes.

Homology

- The equations of stellar structure are homologous – given the solutions for a star of mass M, we can find a solution for another star of mass M' by scaling the other physical variables (provided chemistry is the same).
  - ↳ i.e. assume that for different stars, quantities vary in the same way
  - ↳ this approach may help derive main-sequence shape.
- For two stars of mass M_1, M_2 and radius R_1, R_2:
  - ↳ define the fractional mass contained in a radius $x = \frac{m_1(r_1)}{M_1} = \frac{m_2(r_2)}{M_2}$

$$\left.\begin{aligned} \frac{dm_2}{dr_2} &= \frac{M_2}{M_1}\frac{dm_1}{dr}\cdot = \frac{M_2}{M_1}\frac{R_1}{R_2}\frac{dm_1}{dr_1} = \frac{M_2}{M_1}\frac{R_1}{R_2} 4\pi r_1^2 \rho_1 \\ \frac{dm_2}{dr_2} &= 4\pi r_2^2 \rho_2 = 4\pi\left(\frac{R_2}{R_1}\right)^2 r_1^2 \rho_2 \end{aligned}\right\} \Rightarrow \rho_2(x) = \left(\frac{M_2}{M_1}\right)\left(\frac{R_1}{R_2}\right)^3 \rho_1(x)$$

↳ this gives us the homology transformation for the density.

↳ generally, a particular equation for star 2 can be related to the equation for star 1 either by transforming the derivative or scaling R or M. These approaches must equate, giving a scaling relationship for another quantity.

- The stellar equations (and constitutive relations) in Lagrange coordinates give $r(m)$, $P(m)$, $L(m)$, $T(m)$, $\rho(m)$. These can be replaced with relationships involving x, the fractional mass:

$$\left.\begin{aligned} r &= f_1(x)\cdot R^* \\ P &= f_2(x)\cdot P^* \\ L &= f_3(x)\cdot L^* \\ T &= f_4(x)\cdot T^* \\ \rho &= f_5(x)\cdot \rho^* \end{aligned}\right\}$$

← *quantities are just dimensional coefficients (all f_i dimensionless)

← $0 \le f_i(x) \le 1$ by construction.

← $f_1(1) = 1$, $f_3(1) = 1$

← $f_2(0) = 1$, $f_4(0) = 1$, $f_5(0) = 1$

"direction" of change.

- A stellar equation can thus be rewritten, e.g hydrostatic eq. in Lagrange coordinates $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$

↳ sub $dP = df_2 \cdot P^*$, $dm = dx \cdot M$, $m = x \cdot M$, $r = f_1 \cdot R^*$

$$\Rightarrow \frac{df_2}{dx} = \underbrace{\frac{-x}{4\pi f_1^4}}_{\text{structure (dimensionless)}} \cdot \underbrace{\frac{GM^2}{(R^*)^4 P^*}}_{\text{scaling (dimensional)}}$$

↳ this formulation separates structure and scaling.

↳ considering dimensions, $P^* = GM^2/(R^*)^4$

- Likewise, all of the dimensional coefficients can be expressed in terms of the others - most importantly, M and R^*. This can give us useful relationships between variables.

- The Mass-Luminosity relation can be deduced as $L \propto M^3$, a close approx to the true $M \propto L^{3.5}$. Discrepancy because we ignored convection, treated κ as constant etc.

- The mass-radius relation is $R \propto M^{(\beta-1)/(\beta+3)}$, so for stars burning hydrogen via pp-chain $(\beta = 4)$, $R \propto M^{3/7}$.

- The luminosity-temp relationship comes from plugging the mass-radius and mass-luminosity relations into $L = 4\pi R^2 \sigma T_{eff}^4$

$$\Rightarrow L^{1-2(\beta-1)/3(\beta+3)} \propto T_{eff}^4$$

↳ for pp chain, $\log L = 5.6 \log T_{eff} + c$, which is a reasonable approx to the gradient of the MS on the HR diagram.

To solve Homology Qs:

1. Write out all known equations (e.g structure)
2. Convert to proportionalities

   e.g $\frac{dP}{dr} = -\rho \frac{GM}{r^2} \longrightarrow \frac{P}{R} \propto \rho \frac{M}{R^2}$

3. Eliminate unwanted quantities

Limiting values

- We can find the minimum mass using homology:
  - ↳ minimum core temp required for fusion is $T_{min} \approx 4\times10^6\,K$
  - ↳ we can then scale the solar mass/temp:
    $T^* \propto M^{4/7} \Rightarrow M_{min} = M_\odot\left(\frac{4\times10^6}{1.5\times10^7}\right)^{7/4} \approx 0.1M_\odot$
  - ↳ $L \propto M^3$ so this star would have $L \approx 10^{-3} L_\odot$
- The luminosity of a star is bounded by hydrostatic equilibrium; beyond a certain luminosity, radiation pressure > gravitation
  - ↳ the upper bound is the Eddington limit.
  - ↳ $P_{rad} = \frac{1}{3}aT^4 \Rightarrow \frac{dP_{rad}}{dr} = \frac{4}{3}aT^3 = -\frac{\kappa\rho}{c}\frac{L}{4\pi r^2}$ ← using Eddington equation
  - ↳ but $\left|\frac{dP_{rad}}{dr}\right| < G\frac{M_r\rho}{r^2} \Rightarrow \boxed{L_{edd} = 4\pi c G \frac{M}{\kappa}}$
  - ↳ the Eddington factor quantifies the influence of P_{rad}: $\Gamma_{edd} \equiv \frac{L}{L_{edd}}$

- The maximum mass follows from L_{edd}, using electron scattering as the main source of opacity: $M_{max} = \frac{L_{edd}\,\kappa_{es}}{4\pi c G}\cdot\frac{M_\odot}{L_\odot}$
  - ↳ this gives $M_{max} \approx 200\,M_\odot$, overestimate by ~50%
  - ↳ can be explained by more accurate modelling of κ.

Star Formation

- We do not yet have a predictive theory for star formation (given ICs, predict properties of stars).
- Stars are formed from the interstellar medium (ISM), specifically, Giant Molecular Clouds (GMCs) of H_2 and dust.
  - ↳ typical $M = 10^5 - 10^6 M_\odot$, 10s of pcs.
  - ↳ dust shields molecules from dissociating UV radiation.

START: Gas cloud

1. Free-fall collapse of interstellar cloud
2. Cloud fragmentation, leading to a range of masses
3. Formation of a protostellar core (appears on HR)
4. Accretion of gas via accretion disk.
5. Dissociation of molecules; ionisation of H, He: $H_2 \rightarrow 2H \rightarrow 2H^+ + 2e^-$
6. Pre-main sequence phase

END: Star appears on the Zero-Age Main Sequence (ZAMS)

1. Gravitational collapse

- The Jeans Criterion gives the condition for a cloud to collapse.
- The initial equilibrium is described by the virial theorem:
  - ↳ $2K + U = 0$, collapse if $2K < |U|$ because KE not enough to prevent collapse

↳ $K = \frac{3}{2} NkT = \frac{3}{2} \frac{M_c}{\mu m_H} kT$, $U \simeq -\frac{3}{5} \frac{GM_c^2}{R_c}$

↳ collapse happens if the mass of the cloud exceeds the Jeans mass

$$M_J \simeq \left(\frac{5kT}{G\mu m_H}\right)^{3/2} \left(\frac{3}{4\pi\rho_0}\right)^{1/2}$$

- For a given chemical composition, M_J depends only on temperature and density. We can also write down a Jeans length or Jeans density.
  - ↳ in diffuse H_2 clouds, $M_J \approx 30\,000\, M_\odot$ (very rare).
  - ↳ but in GMC cores, $M_J \simeq 8 M_\odot$ which is common.
  - ↳ initial equilibria may be perturbed by e.g collisions or supernovae, leading to collapse.
- The Jeans model ignores rotation, velocity gradients, magnetic fields, external pressures.
- The energy released by collapse does not all become thermal (else $T\uparrow$ would stop further collapse).
  - ↳ in the early stages of collapse, KE of particles is radiated away as IR (cloud transparent to IR)
  - ↳ hence early collapse is isothermal so can approx as free-fall.
- The free-fall timescale can be estimated by finding $r(t)$ from $\frac{d^2r}{dt^2} = -\frac{GM_r}{r^2}$, in which case t_{ff} is the time for $r = r_0 \to 0$
  - ↳ multiply $\frac{dr}{dt}$ to turn into 1st order $\frac{dr}{dt} = -\left[\frac{8\pi}{3} G\rho_0 r_0^2 \left(\frac{r_0}{r} - 1\right)\right]^{1/2}$
  - ↳ sub $r = r_0 \cos^2\xi \Rightarrow t_{ff} = \sqrt{\frac{3\pi}{32} \frac{1}{G\rho_0}}$
  - ↳ homologous collapse: all parts of the cloud collapse in same t_{ff}.

2. Cloud fragmentation

- Any initial density inhomogeneities may cause regions of the GMC to collapse locally.
- Fragmentation stops when the isothermal assumption fails – the opacity prevents radiation, so the gas heats up and resists further collapse. We can make the approximation of adiabatic collapse.
  - ↳ $P = K\rho^\gamma = \rho kT/\mu m_H \Rightarrow T \propto \rho^{\gamma - 1}$
  - ↳ sub into expression for Jeans mass to get $M_J \propto \rho^{(3\gamma-4)/2}$
  - ↳ H_2 behaves like monatomic gas (rotation mode requires a lot of energy to excite). $\gamma = 5/3 \Rightarrow M_J \propto \rho^{1/2}$
- Hence when collapse becomes adiabatic, M_J increases with density (unlike isothermal), leading to a minimum fragment mass to avoid collapse.
  - ↳ $\Delta K = \frac{1}{2}|\Delta U| \simeq \frac{3}{10} \frac{GM_J^2}{R_J}$
  - ↳ $L_{ff} = \frac{\Delta E_g}{t_{ff}} \sim G^{3/2}\left(\frac{M_J}{R_J}\right)^{5/2}$
  - ↳ $L_{rad} = 4\pi R^2 e\sigma T^4$, where e is an efficiency factor.
  - ↳ $L_{rad} = L_{ff} \Rightarrow M_{J\,min} = 0.03 \left(\frac{T^{1/4}}{e^{1/2}\mu^{9/4}}\right) M_\odot$
- Fragmentation stops when fragments are approx. solar-mass.

3-5. Protostars, Accretion, Dissociation/Ionisation

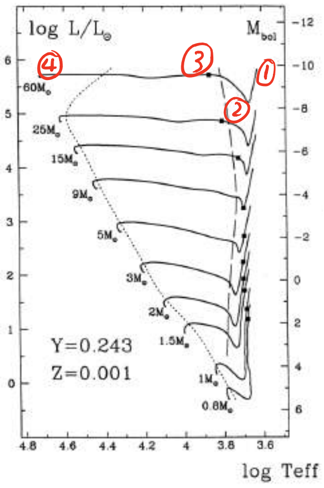
- At some point during collapse the cloud core becomes opaque → hot. The core is in near-hydrostatic eq. (a protostar).
- The outer gas cloud continues free-falling, forming an accretion disk (due to angular momentum).
  - ↳ the luminosity of the protostar comes from the GPE of accretion disks: $L \sim L_{acc} = \frac{1}{2} \frac{GM\dot{M}}{R}$ ← mass accretion rate.
  - ↳ $t_{ff} \ll$ Kelvin-Helmholtz timescale, so core heats up adiabatically.
- As T approaches $\sim 2000\,K$, the energy from contraction now goes to dissociating H_2. Lack of pressure leads to secondary collapse, until H_2 completely dissociated.
  - ↳ this process repeats at ionisation energies ($T \sim 10^4 K$)
  - ↳ after ionisation, the protostar is in hydrostatic eq at a much-reduced radius R_p.
  - ↳ estimate R_p by equating ΔE_g to the sum of dissociation/ionisation energies.

$$\Delta E_g = \frac{3}{10}\frac{GM^2}{R_p} \simeq \frac{M}{m_H}\underbrace{\left(\frac{X}{2}\chi_{H_2} + X\chi_H + \frac{Y}{4}\chi_{He}\right)}_{=\chi}$$

↳ $$R_p \simeq \frac{3}{10}\frac{GMm_H}{\chi}$$

- The temp of the protostar can be estimated via the virial theorem as $\langle T_p \rangle = \frac{2}{3}\frac{\mu}{k}\chi \simeq 8\times10^4 K$. Independent of mass, and far too low for fusion. High opacity (due to H^-) so convective transport.

6. Pre-main sequence



- The right far-right of the HR diagram contains a forbidden region in which temperatures are too low for luminosity to be transported out.
- Pre-main sequence stars follow Hayashi tracks

1. Star is luminous due to energy from collapse. As it contracts, luminosity decreases because opacity is still high and star is convective
2. At a particular temperature, opacity starts decreasing via Kramer's law $\langle\kappa\rangle \propto T^{-3.5}$. A radiative core develops, causing luminosity to increase
3. Contraction continues until the core is hot enough for fusion. Several nucler reactions temporarily create enough pressure to halt contraction, e.g deuterium or lithium burning.
4. Eventually, p-p fusion equilibriates with collapse, leading to a stable zero-age main sequence (ZAMS) star.

- The timescale for 1 → 4 is the Kelvin-Helmholtz timescale: slowest for small R, L.

Objects associated with star formation

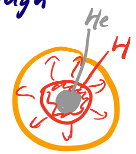
- T Tauri stars are PMS objects lying on Hayashi tracks
  - ↳ luminosity varying on order of days due to accretion
  - ↳ high IR luminosity due to surrounding dust.
  - ↳ fast rotators and purely convective, so high level of activity (e.g flares, X-ray emission).
- Herbig-Haro objects are excitations in the interstellar medium associated with the jets from T Tauri stars.
- OB associations are groups of young O, B main seq. stars that are not gravitationally bound, eventually dispersing.
- Starbursts are intense periods of star formation, which may result in the formation of superclusters.
- O, B stars have high T_{eff} so radiate photons that can ionise H. This results in a H_{II} region ($H_{II} \equiv H^+$)
  - ↳ estimate size of H_{II} region by considering steady state where ionisation rate = recombination rate.
  - ↳ $R_{rec} = \alpha(T) n_H n_e$, $\alpha(T)$ is the recombination coefficient.
  - ↳ the star releases Q_* ionising photons per second, so the Strömgren radius (radius of H_{II} region) satisfies
    $$Q_* = R_{rec} \frac{4}{3}\pi r^3_{H_{II}} \Rightarrow r_{H_{II}} = \left(\frac{3Q_*}{4\pi\alpha}\right)^{1/3} n_H^{-2/3}$$
  - ↳ during recombination, the resulting H_I has an excited electron. The $n=3 \to 2$ transition gives H_{II} regions their red colour.
- Very massive stars ($\sim 20 M_\odot$) have a significant solar wind and highly energetic supernovae, which can disperse GMCs. Hence it is believed that the large stars form last.

Initial mass function

- The distribution of ZAMS stellar masses is described by the initial mass function (IMF), which can be deduced from the present day mass function (PDMF) provided we have an evolution model.
- A simple IMF model is a power law $N(M) = kM^{-\alpha}$, where $N(M)dM$ is the num density of stars with mass $M \in [M, M+dM]$
- Other models stitch together power laws.
- Currently unknown whether IMF is universal or dependent on local conditions e.g metallicity.
- IMF is important when considering the dynamics of galaxies.

Evolution on the Main Sequence

- The evolution of MS stars depends on mass and initial chemistry.
- The MS has intrinsic width (even after accounting for errors) because stars evolve while on the MS.
- As H → He, the mean molecular weight increases. Because $P = \frac{\rho k T}{\mu m_H}$, ρ and T must ↑ to keep P const (to support star)
  - ↳ $\varepsilon_{pp} \propto X^2 \rho T^4$ so this results in the star being more luminous
  - ↳ star moves up/left on HR diagram.
- For a $1 M_\odot$ star, the He core is initially not hot enough for He fusion:



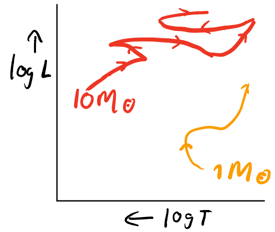
  - ↳ $L_{core} \approx 0$; all luminosity produced by H 'outer core'
  - ↳ by Eddington eq, $L \approx 0 \Rightarrow \frac{dT}{dr} \approx 0$
  - ↳ the pressure gradient is purely provided by the density gradient

$$\frac{dP}{dr} = \cancel{\frac{\rho k}{\mu m_H}\frac{dT}{dr}} + \frac{kT}{\mu m_H}\frac{d\rho}{dr} = -\rho g$$

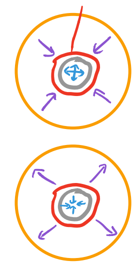
- The increasing temperature now causes the surface of the star to expand, lowering T_{eff} at constant L. Bends to right on HR.
- The pressure gradient from an inert isothermal core can only support so much external mass, so there is an upper bound on the core mass.
  - ↳ beyond the Schönberg-Chandrasekhar limit ($M_{core} > 0.1 M$), pressure support is insufficient so core collapses on Kelvin-Helmholtz timescale.

Massive stars

- Unlike solar mass stars, massive stars have convective cores. This keeps the composition in the core uniform (no inert He inner core).
- The opacity in the core is mostly due to electron scattering.
  - ↳ $\kappa_{es} = \kappa_{0,es} \underbrace{\frac{1}{\mu_e}}_{e^- \text{ per nucleon}} \approx \kappa_{0,es} \cdot \frac{1}{2}(1+X)$
  - ↳ H → He produces e^+, annihilating e^- and reducing κ_{es}
  - ↳ lower κ means that radiation can transport energy effectively, so less convection needed ⇒ core shrinks (different to $M_\odot$ star)
- In massive stars, radiation pressure causes outer layers to expand more rapidly than for $M_\odot$ stars, so throughout their whole MS life, T_{eff} decreases.
- As H is exhausted, the whole star contracts to maintain energy prod. by increasing core temp, producing a left hook on the HR diagram.



The Mirror Principle



- For a star with a shell-burning source, if the inner core contracts, the outer star expands and vice versa.
- Not a physical law; empirical/simulation.
- Energy cons. and virial theorem ⇒ both U and K are individually conserved

$$\langle U \rangle + \langle K \rangle = \text{const} \qquad \langle U \rangle + 2\langle K \rangle = \text{const}$$

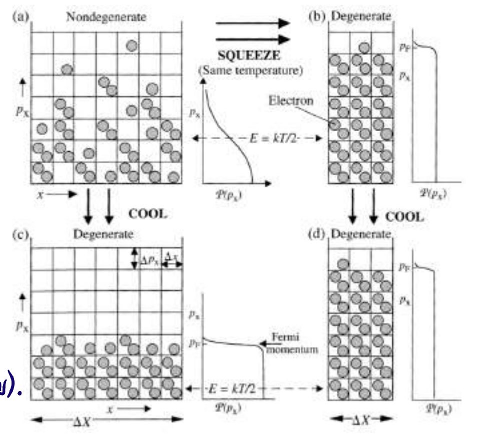
↳ assume that the core mass dominates: $|U| \approx \frac{GM_c^2}{R_c} + \frac{GM_c M_{env}}{R}$

↳ $\frac{d|U|}{dt} = 0 \Rightarrow \frac{dR}{dR_c} = -\left(\frac{M_c}{M_{env}}\right)\left(\frac{R}{R_c}\right)^2$

↳ derivation depends on changes happening on timescales much shorter than Kelvin-Helmholtz timescale.

Electron degeneracy pressure

- In low mass stars ($\leq 2M_\odot$), electron degeneracy is another source of pressure so the core mass may exceed the Schönberg-Chandrasekhar limit.
- Due to the Pauli exclusion principle, only two electrons can occupy a box of volume h^3 in phase space.
- Degenerate gases have a very different distr. to Maxwell-Boltzmann
- Pauli exclusion means that e^- occupy high energy states because there are no free low-energy states.
- Degenerate gases lose temp dependence; pressure only depends on density via $P \propto \rho^{5/3}$
- Temp. can only be measured w.r.t non-degenerate particles (e.g protons).



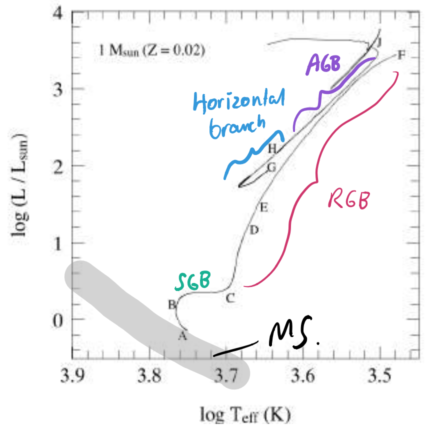
Low mass stars

- $t_{ms} \propto M^{-2.5}$, so for $M < 0.85 M_\odot$ the lifetime is greater than the current age of the universe so none have evolved off the MS.
- Stars with $M \leq 0.3 M_\odot$ are fully convective and fuse/mix until all $H \to He$.
- $M \leq 0.085 M_\odot$ defines brown dwarfs. Too small/cold for H fusion but emit IR by deuterium burning:

$$^1_1H + ^2_1H \rightarrow ^3_2He + \gamma$$

- Objects with no fusion reactions are planets.

Post-MS Evolution for ~$M_\odot$ stars



(A) The $1M_\odot$ star has reached the end of its MS life. It is burning H in a shell around an inert He core.

(B) As H -> He, the core becomes more massive and contracts, leading to expanding outer layers. Energy production/transport is the same so $T_{eff}\downarrow$ to compensate. This defines the sub-giant branch (SGB), which lasts for ~2 Gy.

(C) When T_{eff} low enough, opacity increases due to H^- ions.
↳ the star becomes fully convective and can transport much more energy out.
↳ $L\uparrow$ and star enters the red-giant branch (RGB)

The RGB (C) → (F)

- The RGB is the Hayashi track in reverse (fully convective stars)
- $L\uparrow$ as more H converted to He. The process accelerates because core contraction leads to:
  $\rho\uparrow$ in the H shell → fusion more efficient → $L\uparrow$
- Core density >> outer layer density, so the efficiency of shell burning depends only on core mass (steeply): $L \approx 2\times10^5\left(\frac{M_c}{M_\odot}\right)^6 L_\odot$
- Hence evolutionary paths for a wide mass range converge to the relatively narrow RGB.
- (C)→(F) lasts only 0.5 Gy, at the end of which the degenerate He core has $M\sim0.5M_\odot$ and He fusion can begin.
- The RGB depends on metallicity: higher metallicity → higher opacity ($\kappa \sim T_{eff}^{\,9}$ at these temps) → photosphere further from core (at constant mass) → lower T_{eff}.
- Red giants lose mass because there is weak gravity at the surface but a large photon flux.
  ↳ grains of solid particles are ejected and become part of the interstellar dust.
  ↳ a solar mass star will lose ~30% of its mass as stellar wind.
- During (C)→(D), convection transports material from the core to the surface - dredge-up.
  ↳ Li is fragile, so Li abundance on the surface indicates age.
  ↳ N abundance increases due to C->N equilibrium (during CNO).

The Helium Flash (F)

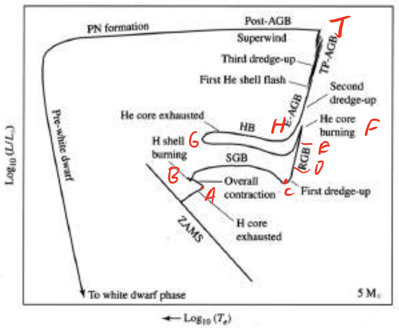
- When $T > 10^8 K$, He nuclei can overcome the Coulomb barrier by tunnelling, so He fusion by the triple-alpha process begins.
- He fusion raises T_{core}, but because the core is degenerate pressure (and density) stay constant.
  - ↳ $\varepsilon_{3\alpha} \propto Y^3 \rho^2 T^{40}$, so there is positive feedback, causing energy production to rapidly increase
  - ↳ during this Helium flash, the core may have local luminosity $10^{10} L_\odot$ for a few seconds, but this is absorbed by the outer layers.
- When $T \simeq 3 \times 10^8 K$, the degeneracy is lifted. The core then expands and cools, leading to a reduction in energy generation, until we reach hydrostatic equilibrium.

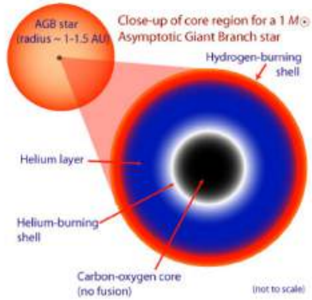
Horizontal branch (G) → (H)

- After the He flash, the star produces energy from both He fusion in the core and H-burning from the shell.
- The core has expanded (compared to prior degenerate state) so the outer envelope contracts. Star less luminous than it was before the He flash, and settles on the horizontal branch (HB)
- But L still high compared to MS, so HB only lasts 0.1 Gy.
- L is independent of total mass because it depends on the He core, which has mass $\sim 0.5 M_\odot$ in all low-mass stars.
- The shape of the HB depends on metallicity
  - ↳ solar metallicities result in a red clump
  - ↳ higher metallicities result in a spread over red/blue
- However, there is an unknown 'second parameter' that explains HB shapes.

Asymptotic Giant Branch (AGB) (H) → (J)

- For $M < 8 M_\odot$, C and O cannot be fused into heavier nuclei
- However, core contraction generates enough heat to start fusing the He shell. This results in behaviour similar to the RGB, which was dominated by H shell-burning.
  - ↳ core contraction → outer layer expansion → $T_{eff} \downarrow$
  - ↳ moves ↗ on H-R diagram, towards the same asymptote as for the RGB.





- AGB is unstable because there is both H/He shell burning: star alternates between them, leading to thermal pulsation (period ~ 1000 yr)

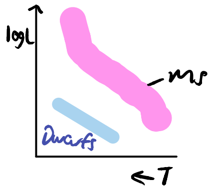
- Because κ has increased, convection becomes more important so there is a second dredge-up, increasing He and N content in the envelope.
- C and O in AGB atmospheres become CO, with excess atoms forming TiO, H_2O, C_nH_n, silicates. Impt source of interstellar dust
- The intershell region is rich in free neutrons, leading to active nucleosynthesis by the s-process. This is evidenced by the presence of technetium, which has no stable isotopes
- On the AGB, mass loss accelerates from $\dot{M} = 10^{-8} M_\odot/yr \rightarrow 10^{-4} M_\odot/yr$
  ↳ stellar wind is $f(L)$ and L is a function of core mass, so as shells burn, more mass is lost as wind.
  ↳ mass loss accelerates until the convective envelope cannot be sustained. It contracts and a new radiative equilibrium is established. The star has now left the AGB
  ↳ this contraction occurs at constant L (determined by core), so $T_{eff} \uparrow$.

Planetary nebulae

- High T_{eff} → luminosity in UV spectrum
  ↳ UV couples strongly with envelope and ionises surrounding dust, creating a H_{II} region
  ↳ these are planetary nebulae (PNs). Called 'planetary' because they have finite extent on the night sky so don't twinkle
- The spectra of PNs are near-discrete emission lines:
  ↳ strongest lines from H/He recombination
  ↳ there are also lines corresponding to collisional excitation
  ↳ lines make it easy to determine radial velocity

White Dwarfs

- White dwarfs are the remnants of low mass stars
  - ↳ occupy a narrow band on the HR diagram, but come in many colours
  - ↳ core mostly He, C, O, with proportions depending on mass.
    - ↳ stars with $M \lesssim 0.5 M_\odot$ will lead to He dwarfs (but these haven't yet formed)
  - ↳ stars with $M \simeq 1-8 M_\odot$ leave CO dwarfs with $M \approx 0.6 M_\odot$

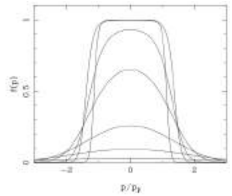


- High surface gravity leads to stratification, so the surface spectrum is very pure (either H or He).
  - ↳ classified as DA if H-dominated (~80%)
  - ↳ DB if He-dominated (~16%)
  - ↳ DC if continuum spectrum; DQ if C present, DZ if metals present.

Electron degeneracy pressure

- Goal is to derive pressure as a function of density.
- Consider the electron states in a box with $V = L^3$: $k_x = \frac{2\pi n_x}{L}$
  - ↳ state density in k-space: $dN = g\left(\frac{L}{2\pi}\right)^3 d^3k$, g is spin degen.
  - ↳ de Broglie to transform to p-space: $dN = g\left(\frac{L}{2\pi\hbar}\right)^3 d^3p$
  - ↳ convert to number density of particles (per unit volume) using the occupation number $f(p)$: $dn = \frac{g}{(2\pi\hbar)^3} f(p)\, d^3p$

- For fermions, Pauli Exclusion $\Rightarrow f(p) \leq 1$ (at most one electron can be in a given momentum state, excl. spin degen).
- From the M-B distr, $dn = \frac{n}{(2\pi m k T)^{3/2}} \exp(-p^2/2mkT)\, d^3p$
  - ↳ there is some critical density where M-B yields $f(p) > 1$ at $p=0$ (peak of distr): $n_{crit} = \frac{g}{\hbar^3}\left(\frac{mkT}{2\pi}\right)^{3/2}$
  - ↳ classical M-B thus breaks down when $n > n_{crit}$
  - ↳ for a fixed density, the classical regime breaks down as $T \to 0$.
- The total number density of particles is: $n = \frac{g}{(2\pi\hbar)^3}\int f(p)\, d^3p$
  - ↳ but in the limit $T \to 0$, states are occupied only up to the Fermi momentum p_F



  - ↳ $n = \frac{g}{(2\pi\hbar)^3}\int_0^{p_F} d^3p = \frac{g}{(2\pi\hbar)^3}\cdot\frac{4}{3}\pi p_F^3$
  - ↳ as $n \uparrow$, $p_F \uparrow$ since lower states fill up.
- Pressure is the flux of momentum: $P_e \simeq p_x n_e v_x$
  - ↳ $dP_x = p_x v_x\, dn_{e,x} \Rightarrow P = \frac{g}{(2\pi\hbar)^3}\int p_x v_x f(p)\, d^3p$
  - ↳ use symmetry and integrate in spherical polars
    $p_x v_x = \frac{1}{3}\underline{p}\cdot\underline{v} \Rightarrow P_e = \frac{g}{3}\frac{1}{(2\pi\hbar)^3}\int_0^\infty \underline{p}\cdot\underline{v}\, f(p)\, 4\pi p^2 dp$
- For non-relativistic electrons, $\underline{p}\cdot\underline{v} = p^2/m_e$. In the $T \to 0$ limit, integrate up to p_F: $P_e = \frac{4\pi g}{3(2\pi\hbar)^3}\int_0^{p_F}\left(\frac{p^2}{m_e}\right)p^2 dp \propto p_F^5$
  - ↳ but $n \propto p_F^3 \Rightarrow p_F \propto \rho_e^{1/3}$
  - ↳ hence $\boxed{P_e = K_1 \rho_e^{5/3}}$, $K_1 = \frac{\pi^2\hbar^2}{5 m_e} 8^{1/3}\left(\frac{6}{g\pi}\right)^{2/3}$
- At high densities, p_F can reach relativistic values
  - ↳ $v = c$, $\underline{p}\cdot\underline{v} = pc \Rightarrow P_e \propto p_F^4 \Rightarrow \boxed{P_e = K_2 \rho_e^{4/3}}$
  - ↳ in both cases, degen. pressure is indep. of temperature.

White Dwarf mass

energy per mode

- The energy density of a degen. gas is $U = \frac{g}{(2\pi\hbar)^3}\int_0^\infty \varepsilon(p)\, f(p)\cdot 4\pi p^2 dp$
- In the relativistic $T \to 0$ case, $\varepsilon(p) = pc$, $f(p) = \mathbb{1}(|p| < p_F)$

  $\Rightarrow U_e = \frac{g}{(2\pi\hbar)^3}\cdot \pi c p_F^4 \propto n_e^{4/3}$

  ↳ the total KE is then $E_K \propto U_e V \propto n_e^{4/3} V \propto M^{4/3}/R$

  ↳ $E_{tot} = E_K + E_P = \frac{AM^{4/3} - BM^2}{R}$

  ↳ the critical mass is such that the two terms are equal so $E_{tot} = 0$
- $AM^{4/3} = BM^2$ gives the Chandrasekhar limit, $M_{Ch} = 1.44 M_\odot$

  For greater masses, the binding energy increases as the star shrinks, leading to unstoppable grav. collapse.
- In the non relativistic case, $E_K = CM^{5/3}/R^2$, $E_P = -BM^2/R$.

  ✡ ↳ equilibrium radius is given by $dE_{tot}/dr = 0 \Rightarrow R = \frac{2C}{B} M^{-1/3}$

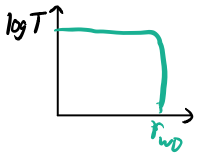
  ↳ $V \propto R^3 \Rightarrow \boxed{M_{WD} V_{WD} = \text{const}}$
- More massive WDs are smaller: need electrons to be more closely confined to support more mass.

White Dwarf Ageing

- Because most electron states are occupied, degenerate e^- can travel far without colliding → e^- conduction is the dominant energy transport mechanism.

  ↳ high efficiency ⇒ isothermal core

  ↳ thin insulating layer at surface; steep T gradient



- Estimate T using the virial theorem: $E_K = \frac{3}{10}\frac{GM^2}{R} = \frac{3}{2}NkT$

  ↳ for a He dwarf, there are $\frac{M}{4m_p}$ nucleons and $\frac{M}{2m_p}$ e^-

  $\Rightarrow E_K = \frac{9}{8}\frac{M}{m_p}kT$

  ↳ gives $T \sim 10^9\,K$ (hot!), radiating ionising X rays
- Cooling in WD comes exclusively from nucleon gas (degen. e^- cannot cool)

  ↳ est. cooling rate using $L \approx dE_K/dt$

  $\Rightarrow 4\pi R^2 \sigma T^4 = \frac{3}{8}\frac{Mk}{m_p}\frac{dT}{dt}$ ← use $E_K = \frac{3}{4}\frac{M}{m_p}kT$, excl e^-

  ↳ this is an upper bound on the rate, neglecting the insulating surface.

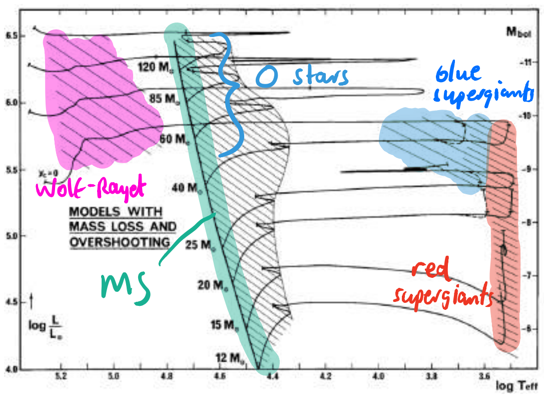
  ↳ $\tau_{cool} = 3\times 10^9\,\mathrm{yr}\,\left(\frac{T}{10^3 K}\right)^{-3}$, i.e. several Gyr for typical WD.

  ↳ WDs actually crystallise, releasing latent heat that further slows cooling.

  ↳ low rate of cooling explains why WDs are white.

Post-MS Evolution: Massive Stars

- Massive stars ($M > 8M_\odot$) can burn C, O in their cores
  ↳ for $M \gtrsim 11 M_\odot$, core temps high enough to fuse up to Fe.
  ↳ mass loss is important at all stages for massive stars. For $M \gtrsim 30 M_\odot$, timescale for mass loss ≤ nuclear burning.
- Because of their large cores, massive stars are overluminous for their masses.



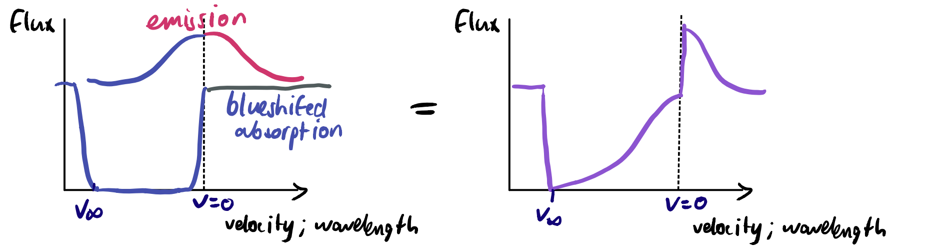
- Massive stars switch between phases of core exhaustion and core ignition, moving left/right on H-R diagram:
  core exhaustion → core shrinks → outer layer expands → $T_{eff} \downarrow$
- Very massive stars ($M \gtrsim 40 M_\odot$) lose most of their envelope as stellar wind, exposing the helium core – Wolf-Rayet (WR) stars
  ↳ WR stars have strong emission lines from the extended gaseous envelopes (rather than absorption lines we'd see from smaller stars)
  ↳ WN WR have strong He, N lines
  ↳ WC/WO WRs have stron He, C/O lines

- Conti's proposed evolutionary scenario:

$M \leq 15 M_\odot$	MS(OB) → RSG (→ BSG → RSG) → SNII
$M = 15\text{-}25 M_\odot$	MS(O) → BSG → RSG → SNII
$M = 25\text{-}40 M_\odot$	MS (O) → BSG → RSG → WNL → WNE → WC → SNIb
$M \gtrsim 40 M_\odot$	MS(O) → BSG → LBV → WNL → WNE → WC → SNIb

Stellar winds

- The solar wind can be estimated as $\dot{M} = n M_H v\, 4\pi d^2$, roughly $10^{-14} M_\odot/yr$. Massive stars have much higher fractional loss rates.
- There is direct evidence for this mass loss: P Cygni line profiles.
  ↳ lines in UV region corresponding to highly ionised species
  ↳ line profile has a mixture of absorption and emission
  ↳ but the absorption line is blueshifted because the stellar atmosphere rapidly expands outwards.

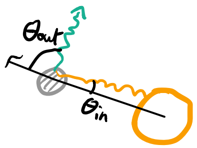


- V_∞ is the terminal velocity of the outflow; can be up to $\sim 3000\,km\,s^{-1}$

- Provided the line is not saturated, we can deduce the ion column densities, which can tell us relative chem. abundances on stars.
- The emission portion of the profile tells us the shape of the velocity field $v(r)$.

Modelling stellar winds

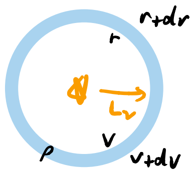
- Model a homogeneous, time-independent and spherically symmetric stellar wind; mostly reasonable, but inhomogeneity (clumping) is impt.
- Momentum is transferred from stellar radiation to the gas.
  - ↳ an element of the wind absorbs photons from the star then re-emits



  - ↳ net radial momentum transfer:

$$\Delta P_{radial} = \frac{h}{c}\left(\nu_{in}\cos\theta_{in} - \nu_{out}\cos\theta_{out}\right)$$

  - ↳ $\langle\cos\theta_{out}\rangle = 0$ (isotropic emission); $\langle\cos\theta_{in}\rangle \approx 1$ because all incident photons are from star $\Rightarrow \langle\Delta P_r\rangle = \frac{h\nu_{in}}{c}$
- Consider a shell of gas around the star



  - ↳ shell mass is $4\pi r^2 \rho\, dr$
  - ↳ lines at ν_i correspond to observed ν_{obs} (difference due to Doppler shift):

$$\nu_i = \nu_{obs} - \frac{\nu_i}{c} v \quad \leftarrow \text{inner surface of shell}$$

$$\nu_i = \nu_{obs} + d\nu_{obs} - \frac{\nu_i}{c}(v + dv) \quad \leftarrow \text{outer surface}$$

$$\Rightarrow d\nu_{obs} = \frac{\nu_i}{c} dv$$

  - ↳ shell acceleration for a transition i is: $g^i_{rad} = \frac{\Delta P}{\Delta t \Delta m}$
  - ↳ num photons per unit time is $\frac{N_\nu}{\Delta t} = \frac{\Delta(E_\nu / h\nu)}{\Delta t} = \frac{L_\nu \Delta\nu_{obs}}{h\nu_{obs}}$

$$g^i_{rad} = \frac{N_\nu \langle\Delta P_r\rangle}{\Delta t \Delta m} = \frac{L_\nu \Delta\nu_{obs}}{h\nu_{obs}} \cdot \frac{h\nu_{obs}}{c} \cdot \frac{1}{\Delta m} = \frac{L_\nu \nu_i}{c^2}\frac{dv}{dr}\frac{1}{4\pi r^2 \rho}$$

  - ↳ shell acceleration depends on the velocity gradient: the larger the range of velocities, the greater the num of interacting photons
- To find the total acceleration we need to sum over all transitions.
  - ↳ the probability of a given transition is related to the opacity
  - ↳ making several approximations, $g^{tot}_{rad} = C\frac{L}{4\pi r^2}\left(\frac{1}{\rho}\frac{dv}{dr}\right)^\alpha$, where $\alpha \approx 2/3$ observationally.
- The properties of the stellar wind can be deduced by solving the structure equations.
  - ↳ result is $\dot{M} \propto L^{1/\alpha}[M(1-\Gamma)]^{1-1/\alpha}$
  - ↳ $\Gamma = \frac{\kappa_{es} L}{4\pi c G M}$ is the Eddington factor (surface gravity reduced by radiation pressure)
  - ↳ relate to line profile parameters:

$$v(r) = v_\infty\left(1 - \frac{R_*}{r}\right)^{1/2} \qquad v_\infty = \left(\frac{\alpha}{1-\alpha}\right)^{1/2}\left(\frac{2GM(1-\Gamma)}{R_*}\right)^{1/2}$$

- Increasing metallicity makes shells optically thicker → more momentum trans.

$$\dot{M}_Z = \dot{M}_{Z\odot}\left(\frac{Z}{Z_\odot}\right)^{(1-\alpha)/\alpha}, \quad v_\infty \propto \left(\frac{Z}{Z_\odot}\right)^{-0.15}$$

Supernovae

- For historical reasons, supernovae are classified as type I (no H in spectrum) or type II (H in spectrum)

H?
- yes → II → Plateau in light curve?
  - yes → II-P
  - no → II-L
- no → I → Si II?
  - yes → Ia
  - no → He I?
    - yes → Ib
    - no → Ic

- Type Ia SNs occur in galaxies of all types; II do not occur in elliptical galaxies (older).

  $\Rightarrow$ { Ia come from long-lived, low mass stars; II come from high mass stars.

- Ia SNs are caused by the thermonuclear explosion of a C/O WD that has accreted mass in a binary
  - ↳ ~25% of SNs are type Ia
  - ↳ on average, these are the most luminous
  - ↳ light curves are mostly the same ⇒ we as standard candles.
- II, Ib, Ic SNs are core-collapse SNs: the last stage in the evolution of massive stars.
  - ↳ II / Ib / Ic depends on which shells on the star remain
- All the SNs we've observed have come from $8-17M_\odot$ stars. One hypothesis is that these stars collapse directly into black holes, without rejecting any material.

Core collapse ($M \gtrsim 11 M_\odot$)

- Once the Fe core reaches the Chandrasekhar limit, e^- degen. cannot support it so th core collapses (in less than a second)

  $R_{c,i} \sim 3000\,\mathrm{km} \rightarrow R_{c,f} \sim 20\,\mathrm{km}$

- Estimate energy release using the virial thm:

  $$U_{gr} = -\frac{3}{10}\frac{GM_c^2}{R_{c,i}} + \frac{3}{10}\frac{GM_c^2}{R_{c,f}} \simeq \frac{3}{10}\frac{GM_c^2}{R_{c,f}}$$

  - ↳ typically $U_{gr} \sim 10^{46}\,J$, greater than the binding energy of the star.
  - ↳ mass $\sim 10 M_\odot$ ejected at $\sim 3000\,\mathrm{km\,s^{-1}} \Rightarrow E_{ej} \sim 10^{44}\,J$
  - ↳ $\sim 10^{42}\,J$ released as radiation
  - ↳ all this accounts for ~1% of available U_{gr}; rest is carried away by neutrinos
- Core collapse is a positive feedback loop:
  - ↳ energetic photons photodisintegrate Fe, reducing the pressure ⇒ core contracts
  - ↳ as $T\uparrow$, photons eventually become energetic enough to break He

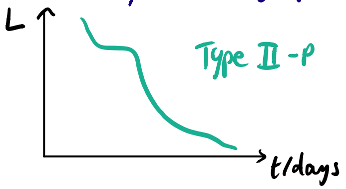
    ${}^4_2He + \gamma \rightarrow 2p^+ + 2n$
  - ↳ when pressure is high enough, inverse β-decay occurs,

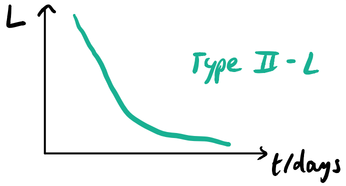
    $p^+ + e^- \rightarrow n + \nu_e$, so e^- degen pressure is lost → COLLAPSE

- At a certain density, nucleon degeneracy instantaneously makes the core incompressible, so the collapse reverses → core bounce
  - ↳ the interaction between core bounce and freefalling material creates shockwaves
  - ↳ it was previously thought that these shockwaves blow off the envelope, but newer models show that the energy instead disintegrates heavy nuclei.
  - ↳ some unknown mechanism later causes the explosion. One theory is that neutrinos are trapped in the core because of the high density. When these escape (~0.1s), the star explodes
- Photodisintegration and inverse β-decay provide free neutrons, which can be captured by the r-process to produce post-Fe elements.
- The core remnant becomes a neutron star or BH, depending on whether its mass is below the Oppenheimer-Volkoff limit.

Light curves of core-collapse SNs.

- When the shockwave reaches the surface there is a bright flash of X-rays, after which luminosity rapidly declines.
- If there is a large H shell, the gas that was ionised experiences recombination, releasing photoelectrons and resulting in a plateau





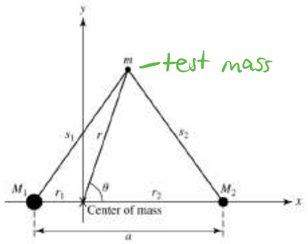
- The latent luminosity is a result of radioactive decay:
  - ↳ the shockwave causes explosive nucleosynthesis of $^{56}_{28}Ni$ from $^{28}_{14}Si$ (timescales too short for β-decay into $^{56}_{26}Fe$)
  - ↳ β^+ releases radiation: $^{56}_{28}Ni \xrightarrow{\tau=6d} {}^{56}_{27}Co \xrightarrow{\tau=77d} {}^{56}_{26}Fe$
  - ↳ hence luminosity ($\propto$ decay rate) decreases exponentially.

Gamma-ray bursts (GRBs)

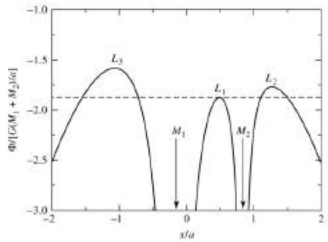
- GRBs are the most energetic astrophysical events (10^7x brighter than the most luminous SNs).
- Short-hard GRBs last <2s, are high freq, and are associated with mergers of neutron stars / BHs
- Long-soft GRBs may be a result of core-collapse SNs of rotating massive stars.
- GRBs are visible out to cosmological distances, giving us info about the early universe.

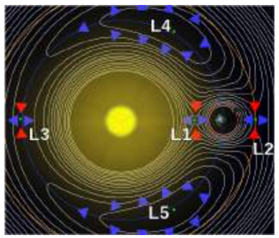
Close Binary Systems

- A close binary star system is one in which the separation is comparable to the size of the stars.
- Work in a corotating CoM frame:
  - ↳ $F_g = -\frac{GMm}{r^2}\hat{r}$ balanced by $F_c = m\omega^2 r\,\hat{r}$
  - ↳ the effective potential is $\Phi = -G\left(\frac{M_1}{s_1} + \frac{M_2}{s_2}\right) - \frac{1}{2}\omega^2 r^2$
  - ↳ orbital freq is $\omega^2 = \frac{G(M_1+M_2)}{a^3}$

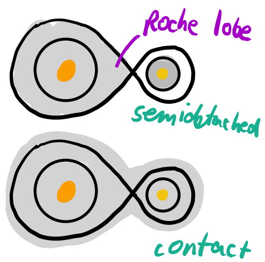


- Lagrangian points have $\frac{d\Phi}{dx} = 0$:
  - ↳ no net force on a test mass
  - ↳ values of x/a are labelled L_n
  - ↳ L_n are all unstable equilibria
  - ↳ the inner Lagrangian point L_1 is particularly important in the evolution of close binaries.





- Systems may expand to fill equipotential surfaces: density/pressure must be constant on equipotentials.
  - ↳ in a semidetached binary, the secondary star fills its equipotential up to L_1
  - ↳ as it expands beyond its Roche lobe, mass transfer from secondary → primary begins.



Mass transfer

- Orbital motion may result in the formation of accretion disks.
  - ↳ hot spot where the mass stream hits the accretion disk.
  - ↳ as mass falls in, its angular momentum must be transported outwards. This is thought to be caused by turbulence-enhanced viscosity.
- As mass is transferred, the separation a may change, causing the period to also change.
- In a simple model, consider a circular binary with constant total mass $M = M_1 + M_2$
  - ↳ $L = \mu\sqrt{GMa}$, $\mu = \frac{M_1 M_2}{M_1+M_2}$
  - ↳ conserve L, M $\Rightarrow$ $\dot{L} = 0$, $\dot{M} = 0$

    $\Rightarrow \frac{1}{a}\frac{da}{dt} = 2\dot{M}_1\left(\frac{M_1 - M_2}{M_1 M_2}\right)$
  - ↳ $\omega \propto a^{-3/2}$ $\Rightarrow$ $\frac{1}{\omega}\frac{d\omega}{dt} = -\frac{3}{2}\frac{1}{a}\frac{da}{dt}$

Type Ia Supernovae

- Ia SNs occur when mass transfer to a WD causes it to exceed the Chandrasekhar limit.
- In the single degenerate scenario, before exceeding M_c, T gets high enough for oxygen burning → runaway CO detonation because degen. pressure is indep. of T
- In double degen models, 2 WDs merge. Unknown if this leads to SN or collapses directly to a neutron star.