1A Gravitational Fields

- Force between two masses $F = -\frac{GMm}{r^2}\hat{\underset{\sim}{r}}$
  ↳ no net force on the system.
- The field is the grav. force per unit mass at a point : $F = mg$.
- For a single particle, the field is $g(\underset{\sim}{r}) = \frac{GM}{r^2}\hat{\underset{\sim}{r}}$..
- Fields can be thought of as flux lines, because the number of flux lines through a fixed area varies inverse square.
- Gauss' law relates the net flux out of a closed surface to the enclosed mass: $\oiint_S g \cdot d\underset{\sim}{s} = -4\pi GM$
  ↳ only practically useful for highly symmetrical surfaces, i.e spheres, cylinders or planes.

- The (scalar) gravitational potential is the work done per unit mass in moving a mass from infinity to the point: $\phi(\underset{\sim}{r}) = -\sum_i \frac{GM_i}{r_i}$
- The gravitational PE for a system can be found by adding one mass at a time: $PE = -\frac{1}{2}\sum_i \sum_{j\neq i} \frac{GM_iM_j}{r_{ij}}$
  ↳ for continuous shapes, we can imagine building in layers

Orbits

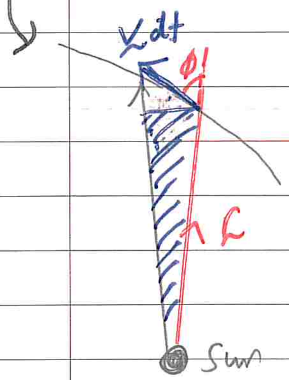
- Consider the motion of many particles in terms of motion of the COM and motion about the COM.
  - linear momentum about COM is zero
  - KE_{total} = KE in COM frame + KE of COM
  - ~~ang~~ $\underset{\sim}{L}_{total}$ = L about COM + L of COM about origin
    ↳ i.e parallel axis thm.
- Orbital motion must be analysed in the COM frame (min KE) such that $E_{total} < 0 \Rightarrow$ bound, $E_{total} > 0 \Rightarrow$ unbound.
  ↳ approx that if one mass is very large, the system COM is the same as that mass' COM $\Rightarrow$ small mass has all KE.

- For a satellite in orbit: $F = \frac{mv^2}{R} = \frac{GMm}{r^2} \Rightarrow KE = \frac{GMm}{2r}$

- The escape velocity is when $K.E = |PE|$ $\therefore$ $v = \sqrt{\frac{2GM}{r}}$
  ↳ this is true regardless of the angle

Kepler's laws

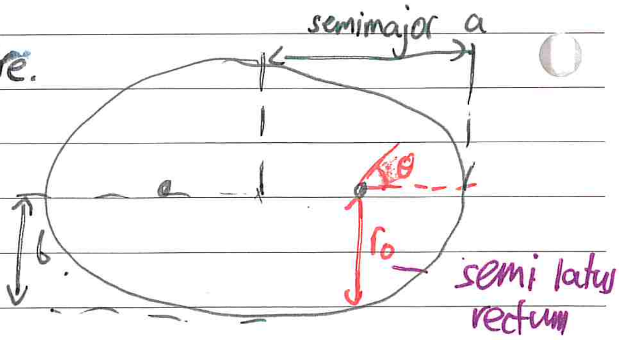
1. The orbit of a planet around the sun is an ellipse with the Sun at one focus
2. The radius vector of a planet sweeps equal areas in equal time
3. $T^2 \propto a^3$, a is the semimajor axis



The area of this triangle $= \frac{1}{2}$ base $\times$ height
$= \frac{1}{2}(v\,dt \sin\phi) \times r$

But $|L| = mrv\sin\phi \Rightarrow \frac{dA}{dt} = \frac{L}{2m}$ (const).

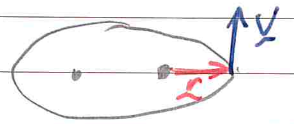
- Ellipses are defined by $r = \frac{r_0}{1+e\cos\theta}$ w.r.t the origin at a focus.
  - or $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for origin at centre.
  - area $= \pi ab$
  - $e = \sqrt{1 - \frac{b^2}{a^2}}$
  - distance from origin to focus $= ea$.



- Orbits can actually follow any conic section:
  - $E_{total} < 0 \Rightarrow$ bound orbit i.e $0 \le e < 1$
  - $E > 0 \Rightarrow$ unbound i.e hyperbola, $e > 1$.
- K3 can be derived from K1 and K2 using:
  $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m}$ then writing $m\dot{v} = -\frac{GMm}{r^2}\hat{r}$.

- We can show that: $\dot{\underset{\sim}{v}} = -\frac{L^2}{m^2 r_0} \cdot \frac{1}{r^2} \hat{\underset{\sim}{r}}$
  - ↳ hence Newton knew that gravity must be a central force.

- In practise, it is easiest to analyse orbits by considering cons L and cons E_{total}.
  - ↳ at the perihelion and aphelion, $v \perp r$ hence $L = mvr$.

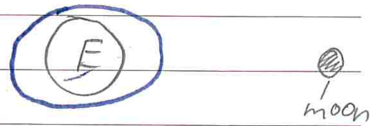


  - ↳ the semi-latus rectum is a function of L:

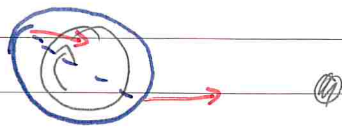
$$r_0 = \frac{L^2}{GMm^2}$$

Tides

- A result of variation in field strength:
  - water on near side pulls away from the earth, which pulls away from the water on the far side.



- In reality, the bulge is dragged by the earth
  - thus, there is a net turning moment that slows down the Earth's rotation



  - effect much greater on the moon, hence the moon's orbital period is now synchronised with rotation.

- Tidal forces can result in heating, e.g Io's orbit involves pull from Jupiter and other moons, resulting in significant heat.

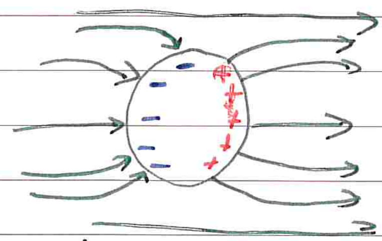
1A Electromagnetism

- Coulomb's law: $F = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{r}$
- The electric field at a point is the force per unit charge.

$$\oint_S \underset{\sim}{E} \cdot d\underset{\sim}{S} = \frac{Q_{enc}}{\varepsilon_0}$$

↳ we can construct Gaussian surfaces as with gravity.

- In the steady state, there can be no E field inside a conductor: charges will reorganise to remove the E field
  - $E \perp$ surface of conductor
  - field lines begin on ⊕ and terminate on ⊖
  - the surface of a conductor will be an equipotential.



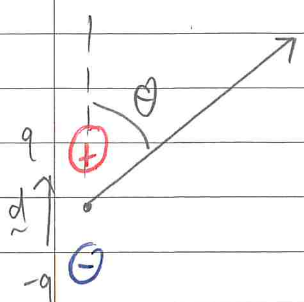
- Thus there can be no field inside, because lines would have to start and end on the inner surface, which is impossible because the surface is an equipotential. ⇒ Faraday cage.

- The field near a surface of a conductor can be approximated as a plane/sphere
  - for a given sphere, $E = \frac{q}{4\pi r^2} = \frac{V}{r}$ i.e $E \propto \frac{1}{\text{radius}}$
  - hence spikes on conductors have huge E fields.

Capacitance

- How much charge is required to increase potential by 1V.
- In general, we just find an expression for V, e.g using Gauss's law then integrating, then $C = Q/V$.
- An isolated sphere has a capacitance: the other plate is at infinity.
- The energy stored by a capacitor is $\frac{1}{2}CV^2$
  - energy density is thus $U_E = \frac{1}{2}\varepsilon_0 E^2$, which applies to general E fields.
  - nonlinear hence can't superpose electrostatic energy density.
  - $\frac{1}{2}\varepsilon_0 E^2$ is also the force per unit area: electrostatic stress.

Dipoles



- Consider a dipole $\underset{\sim}{p} = q\underset{\sim}{d}$

$$\phi(\underset{\sim}{r}) = \frac{q}{4\pi\varepsilon_0|\underset{\sim}{r} - \frac{\underset{\sim}{d}}{2}|} - \frac{q}{4\pi\varepsilon_0|\underset{\sim}{r} + \frac{\underset{\sim}{d}}{2}|}$$

- With $|\underset{\sim}{r}| \gg |\underset{\sim}{d}|$, $|\underset{\sim}{r} \pm \frac{\underset{\sim}{d}}{2}| \approx r \pm \frac{d}{2}\cos\theta$

$$\therefore \phi(\underset{\sim}{r}) = \frac{qd\cos\theta}{4\pi\varepsilon_0(r^2 - d^2\cos^2\theta/4)}$$

$$\Rightarrow \phi(\underset{\sim}{r}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

- Thus it looks like a single charge, except drops off as $1/r^2$
- A dipole placed in a field will experience a couple which acts to align the dipole with the field: $\underset{\sim}{\tau} = \underset{\sim}{p} \times \underset{\sim}{E}$
- Dipoles can be induced in atoms due to electron clouds moving in response to an external field.

Magnetic Fields

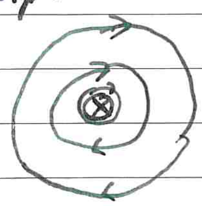
- In addition to the electrostatic force, there will be another force on a charge if that charge is moving.

$$\therefore \underset{\sim}{F} = q\underset{\sim}{E} + q\underset{\sim}{v} \times \underset{\sim}{B}$$

- B is measured in tesla, i.e. $NA^{-1}m^{-1}$.
- Because all B fields form closed loops, $\oint\!\!\oint_S \underset{\sim}{B} \cdot d\underset{\sim}{S} = 0$, and $\nabla \cdot \underset{\sim}{B} = 0$.
- We commonly want to know the flux: $\phi_B = \iint_S \underset{\sim}{B} \cdot d\underset{\sim}{S}$ (Wb)
- A current through a wire produces a B field:

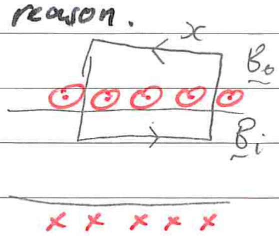
Ampere's law: $\oint \underset{\sim}{B} \cdot d\underset{\sim}{l} = \mu_0 I$ ← $\mu_0 = 4\pi\times10^{-7} NA^{2}$, and is used to define the amp.

↳ using circular symmetry, $B = \frac{\mu_0 I}{2\pi r}$



↳ inside a solid wire, $B \propto r$, which we can find by considering enclosed current.

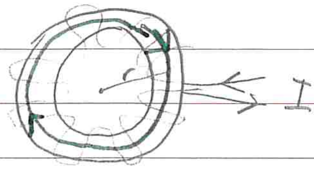
- Consider a solenoid of length l and N turns:
  - no azimuthal component of $\underset{\sim}{B}$ because no enclosed current
  - no longitudinal component outside for the same reason.
  - for a loop near the middle:



$$\oint_c \underset{\sim}{B} \cdot d\underset{\sim}{l} = B_i x - B_o x = \mu_0 I_{enc}$$

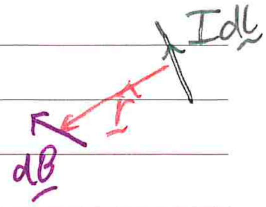
$$\therefore B_i = \mu_0 \frac{N}{l} I$$

- For a toroidal solenoid, $B = \frac{\mu_0 N I}{2\pi r}$
  - ↳ $B=0$ outside the loop.



Biot-Savart Law

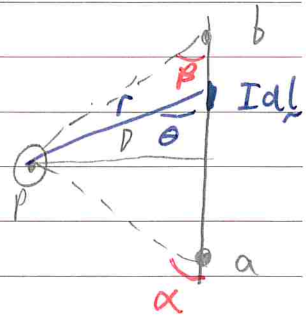
- Similar to Coulomb's law: $d\underset{\sim}{B} = \frac{\mu_0 I d\underset{\sim}{l} \times \hat{\underset{\sim}{r}}}{4\pi r^2}$
  - ↳ current elements are the source of fields
  - ↳ but currents must be part of a complete circuit.



- e.g. for a finite wire:

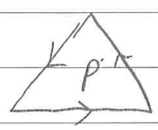
$$B = \int_a^b \frac{\mu_0 I d\underset{\sim}{l} \times \hat{r}}{4\pi r^2} = \int_a^b \frac{\mu_0 I \sin\theta}{4\pi r^2} dl$$

but $\sin\theta = \frac{D}{r}$ and $dl = -\frac{D}{\sin^2\theta} d\theta$

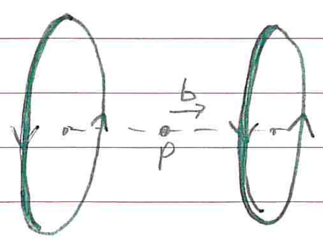


$$\therefore B = \frac{\mu_0 I}{4\pi D}(\cos\beta - \cos\alpha).$$

- This can be used to evaluate the B field due to a coil made of straight segments



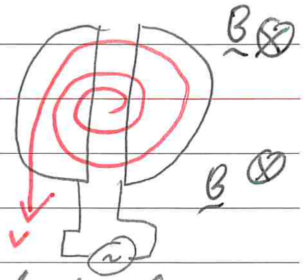
- Applying Biot-Savart to a current loop shows that it acts as a magnetic dipole with strength $I\underset{\sim}{A}$.
- We can create a relatively uniform field between two similar coils – a Helmholtz pair:
  - ↳ the ideal separation can be found by Taylor expanding for a small disturbance.
  - ↳ most uniform when the points of inflection coincide, ie $\Rightarrow$ separation = radius



- Alternatively we can construct a uniform gradient field with a Maxwell pair (opposite loops), using a separation that makes $f^{(3)}(0) = 0$.
- Helmholtz and Maxwell pairs are useful in MRIs.

Motion in fields

- A uniform electric field makes a charged particle move in a parabola.
- A uniform magnetic field causes circular motion: $qvB = \frac{mv^2}{R}$.
- JJ Thomson's experiment uses cross B/E fields to determine the ~~B~~ q/m ratio for an electron.
- The circular motion can be used to accelerate charged particles, as in a cyclotron
  - $f = qB/2\pi m$, i.e. independent of radius
  - the AC freq is set to equal f, so the E field between dees switches at the right time.



- At higher energies, relativistic effects mean that the frequency drops, so must be synchronised – synchrotron.

- A magnetic field exerts a force on a wire: $\underset{\sim}{F} = I d\underset{\sim}{l} \times \underset{\sim}{B}$
- Thus two wires with opposite currents experience an attractive force: $\frac{F}{L} = \frac{\mu_0 I_1 I_2}{2\pi r}$

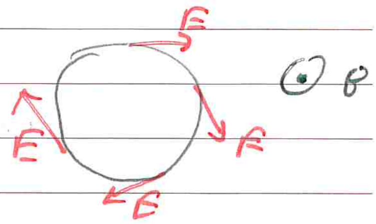
  ↳ this was how the amp was defined.

Electromagnetic Induction

- A change of flux results in an induced emf.
- Faraday's law: magnitude of induced emf is equal to the rate of change of flux through the loop.
- Lenz's law: direction of induced current opposes change that caused it.

$$\mathcal{E} = -\frac{d\phi_B}{dt} = -\frac{d}{dt}\int_S \underset{\sim}{B} \cdot d\underset{\sim}{S}$$

- For uniform B, $\phi = AB\cos\theta$.



- An emf of $\mathcal{E}$ around a loop means that $q\mathcal{E} = \oint \underset{\sim}{F} \cdot d\underset{\sim}{l}$ by definition. This F must be electric since it is velocity-independent
  $\therefore \; \mathcal{E} = \oint \underset{\sim}{E} \cdot d\underset{\sim}{l}$.
- But from Faraday/Lenz, $\mathcal{E} = -\frac{d}{dt}\int_S \underset{\sim}{B} \cdot d\underset{\sim}{S}$
  ↳ if S is constant: $\mathcal{E} = \int_S \nabla \times \underset{\sim}{E} \cdot d\underset{\sim}{S} = -\int_S \frac{\partial B}{\partial t} \cdot d\underset{\sim}{S} \Rightarrow \nabla \times \underset{\sim}{E} = -\frac{\partial B}{\partial t}$.

Inductance

- An inductor (e.g solenoid) will show resistance to changes in current.
- The self-inductance is defined to be $L = \phi_B / I$, and depends on the coil geometry.
- To calculate it, we just need to find ϕ_B. e.g for a solenoid:
  - $B = \mu_0 N/l \, I$
  - the flux linking one turn is BA
  - hence total flux linked is $NBA \Rightarrow L = \mu_0 A \frac{N^2}{l}$.
- This is important because changes in current lead to an induced emf!
  ↳ if the current in an inductor is suddenly zero (e.g circuit break), there will be a huge ~~Em~~ emf, and maybe a spark.
- Inductors store magnetic energy: $W = \frac{1}{2} L I^2$
  ↳ leads to the general result that the energy density of a B field is $U_m = \frac{1}{2}\frac{B^2}{\mu_0}$. Including electric, $U_{tot} = \frac{1}{2}\left[\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right]$

Maxwell's Equations

div { • Gauss's law + div theorem $\Rightarrow$ $\nabla \cdot \underset{\sim}{E} = \frac{\rho(\underset{\sim}{r})}{\varepsilon_0}$

{ • No magnetic monopoles $\Rightarrow$ $\nabla \cdot \underset{\sim}{B} = 0$

Stokes { • Ampere's law $\Rightarrow$ $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}$ $\leftarrow$ current density vector.

{ • Faraday / Lenz $\Rightarrow$ $\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$

• But actually these equations violate conservation of charge, because Ampere's law gives $\nabla \cdot \underset{\sim}{J} = 0$

↳ charge cons requires $\nabla \cdot \underset{\sim}{J} = -\frac{\partial \rho}{\partial t}(\underset{\sim}{r}) = -\varepsilon_0 \nabla \cdot \dot{\underset{\sim}{E}}$

↳ Maxwell proposed adding a fix, adding this as a displacement current $\therefore$ $\nabla \times \underset{\sim}{B} = \mu_0 \left(\underset{\sim}{J} + \varepsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t} \right)$

• These equations imply the existence of transverse EM waves.

- $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{k}$ form a mutually orthogonal set
- energy shared equally between E and B fields.