IA Mechanics

No. 1
Date 9.1.18

Dimensions and Units

- Physical quantities have dimensions: some product of M L T.
- Can be used to guess relationships using $[LHS] = [RHS]$
- If there are too many unknowns, we can form a dimensionless group.

$$\left[\frac{A}{B}\right] = 1 \Rightarrow [A] = [B] \Rightarrow A = f(B)$$ for some dimensionless f.

- ~~Units~~ Quantities can be treated as the product of the value and unit.

Experimental Physics

- Random errors can only be removed by taking more readings.
- Systematic errors cannot be removed by repetition.
- The best estimate of the true value is the sample mean

$$\bar{x} = \frac{1}{n}\sum x_i \qquad s_{n-1} = \sqrt{\frac{1}{n-1}\sum_i^N (x_i - \bar{x})^2}$$

- For n independent measurements, the error in the mean is $s_{n-1}/\sqrt{n}$
  - σ_{mean} should be reported to 1sf unless it starts with a 1 or 2
  - then $\bar{x}$ rounded to same no. of decimals

  e.g. (31.42 ± 0.16) cm $\qquad (12.93 \pm 0.07)\ \Omega$

- To combine independent errors: adding in quadrature

$$Z = f(x, y) \Rightarrow (\Delta Z)^2 = \left(\frac{\partial f}{\partial x}\Delta x\right)^2 + \left(\frac{\partial f}{\partial y}\Delta y\right)^2.$$

special cases: $Z = A^n \Rightarrow \frac{\Delta Z}{Z} = |n|\frac{\Delta A}{A}$

- To reduce time-dependent systematic errors, we can ~~me~~ vary the independent variable in a different order.

· If we are measuring ~~the~~ a continuous attribute of a discrete variable (e.g time of a pendulum swing), we can use the method of exact fractions

↳ get an initial estimate and error, e.g time 5 swings 3x.
↳ time an unknown number of swings
↳ divide by estimated T to get estimated no. of swings
↳ update estimate of T

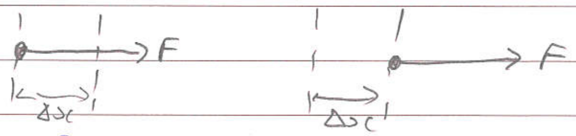
Forces

· Vector quantity: $\vec{F}_{net} = \sum \vec{F}_i$
· In equilibrium, $\vec{F}_{net} = 0$. This applies to any cut we make
· Contact forces (e.g the normal) are described by Newton's 3rd.
· Friction is described as:

$$F_{max} = \mu_s N \quad \text{(static)} \quad \text{or} \quad F_s \leq \mu_s N$$
$$F_d = \mu_d N \quad \text{(dynamic)}$$

· Work is done when the point of application of the force moves in the direction of the force.

$$\Delta W = F \Delta x$$



· More generally: $W_{12} = \int_{\underset{\sim}{r_1}}^{\underset{\sim}{r_2}} \underset{\sim}{F} \cdot d\underset{\sim}{r}$

· The potential energy is the work done to put particles in a particular arrangement, starting from some reference point.

$$F_{int} = -\frac{dU}{dx} \implies \text{equilibrium when } \frac{dU}{dx} = 0$$

- if U is a minimum, stable equilibrium
- else U is at an unstable equilibrium, small Δx leads to force away from equilibrium.

Dynamics

- Equation of motion: $\ddot{\underset{\sim}{x}} = \underset{\sim}{F}/m$ (Newton's 2^{nd}).
- Power is the rate at which work is done:

$$P = \frac{dW}{dt} = \underset{\sim}{F} \cdot \underset{\sim}{v}$$

- If work is done on an object, its speed increases, giving it KE.
- Momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$ or $\underset{\sim}{F} = \frac{d\underset{\sim}{p}}{dt}$
- Conservation of momentum: the total linear momentum of an isolated system is constant.
  - ↳ applies to components
  - ↳ even with external $\underset{\sim}{F}$, sum (internal F) $= 0$.
- In a collision, momentum is conserved
  - ↳ elastic: KE conserved
  - ↳ inelastic: some KE → internal energy.
- In a collision, force is unlikely to be constant. Thus we need to integrate: impulse $= \int F\,dt$ = change in momentum.

Frames of reference

- We can transform frames by adding/subtracting velocity.
- In an inertial frame of reference, Newton's 1^{st} law is valid.
- An instantaneous rest frame makes one object stationary for a moment.
- Although the total KE may be different for different frames, ΔKE will be the same.
- The zero momentum frame is such that the total momentum is zero, found by subtracting V_{zm} from each particle.

$$\underset{\sim}{V}_{zm} = \frac{\underset{\sim}{p}}{\sum_i m_i} = \frac{\sum_i m_i \underset{\sim}{v}_i}{\sum m_i}$$

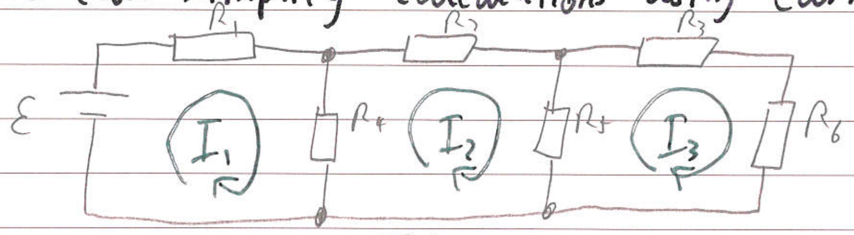
↳ after working out the collision, add back V_{zm} to all particles.

IA Circuits

No. 1
Date 31.12.18

- charge in an electric field experiences $\vec{F} = q\vec{E}$
- In a wire: $I = \frac{dQ}{dt} = nqA\langle v \rangle$ cross sectional area.
- p.d is work done per unit charge, in moving q in $\vec{E}$.
- emf ε is the energy gained per unit charge in a cell.

- ~~Kirchh~~ Kirchhoff's current law: current conserved at junction
- Kirchhoff's voltage law: around any loop, $\sum V_i = 0$ (for cons energy).
- We can simplify calculations using current loops:



$\therefore\ \varepsilon - I_1R_1 - (I_1 - I_2)R_4 = 0, \quad -I_2R_2 - (I_2 - I_3)R_5 - (I_2 - I_1)R_4 = 0$

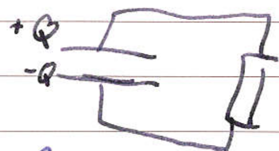
... (3 eq with 3 unknowns).

- For voltage, we can arbitrarily label voltage w.r.t any reference point of our choosing (negative terminal usually).

Capacitors

Defined by $C = \frac{Q}{V}$. $\therefore\ W = \int V dq = \frac{1}{2}CV^2$

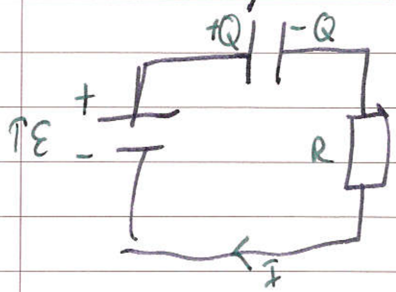
- In an RC circuit with a charged capacitor:



$V = IR = \frac{Q}{C} \quad \therefore\ \frac{dQ}{dt} = \frac{Q}{RC} \Rightarrow Q(t) = Q_0 e^{-\frac{t}{RC}}$

↳ exponential decay with characteristic time $\tau = RC$

- To charge a cap



$\varepsilon = IR + \frac{Q}{C}$

$\Rightarrow Q = C\varepsilon(1 - e^{-\frac{t}{RC}})$

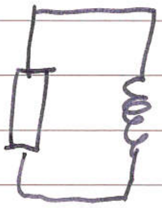
Inductors

$\varepsilon = -\frac{d\phi_B}{dt}$ But we also have $\phi_B = LI$ by definition.

self-inductance.

$\therefore \; \varepsilon = -L\frac{dI}{dt}$

↳ stores energy in the field: $W = \frac{1}{2}LI^2$



In an RL circuit, $-IR - L\frac{dI}{dt} = 0$

$\Rightarrow I(t) = I(0)\, e^{-\frac{tR}{L}}$

i.e exponential decay with characteristic $\tau = \frac{L}{R}$.

IA Oscillating Systems

- A system oscillates if $x(t) = x(t+T)$ for some Period T.
- The frequency of osc. is defined as $\nu = \frac{1}{T}$ (s^{-1})
- The angular freq is $\omega = 2\pi/T$
- For an object to undergo simple harmonic motion (SHM):
  - must be some inertia
  - and a restoring force $\propto$ (− displacement)

  $F = -kx = ma$ for a spring $\Rightarrow$ $\ddot{x} = -\frac{k}{m}x$
- If we sub $x = a_0 \cos\omega t$ we find $\omega = \sqrt{\frac{k}{m}}$

- The general equation for undamped SHM is $\ddot{x} + \omega_0^2 x = 0$
- Solved by:

$$\left.\begin{aligned} x(t) &= a_0\cos(\omega t + \phi) \\ x(t) &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \end{aligned}\right\} \text{2 unknowns}$$

- SHM is unique among oscs because freq is independent to amplitude.
- The velocity and acceleration can be found by differentiating:

  $\dot{x}(t) = -\omega_0 a_0 \sin(\omega_0 t + \phi)$ ($\frac{1}{4}$ cycle ahead)

  $\ddot{x}(t) = -\omega_0^2 a_0 \cos(\omega_0 t + \phi)$. ($\frac{1}{2}$ cycle ahead).

  we then see that $v_{max} = -\omega_0 a_0$ and $a_{max} = -\omega_0^2 a_0$

- We can instead analyse systems in terms of energy. For a spring:

  $KE = \frac{1}{2}m\dot{x}^2$ $\quad PE = \int_0^x kX\,dX = \frac{1}{2}kx^2$

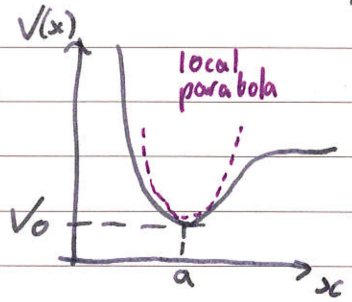
$$\Rightarrow \quad \begin{aligned} KE &= \tfrac{1}{2}k a_0^2 \sin^2(\omega_0 t + \phi) \\ PE &= \tfrac{1}{2}k\, a_0^2 \cos^2(\omega_0 t + \phi) \\ E_{total} &= \tfrac{1}{2}k a_0^2 \end{aligned}$$

- Thus the energy terms oscillate twice as fast (e.g write $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$)
- Because $\langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle = 1/2$, we know that $\langle KE \rangle = \langle PE \rangle = \frac{1}{4}ka_0^2$

- We can instead derive SHM using the conservation of energy

  $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \Rightarrow \dot{E} = m\dot{x}\ddot{x} + kx\dot{x} \Rightarrow \ddot{x} + \frac{k}{m}x = 0.$

- Thus, any time a particle moves in a quadratic PE, it will undergo SHM.
- This can be applied to more complex potentials:



Taylor expanding about $x=a$:

$$V(a+\Delta x) = V(a) + V'(a)\Delta x + \frac{V''(a)}{2}(\Delta x)^2 + \dots$$

At the minimum, $V'(a)=0$

$$\therefore \quad V(a+\Delta x) \approx V(a) + \frac{1}{2}V''(a)(\Delta x)^2$$

- Small peturbations can be approxed by quadratic potentials
  ↳ modeled with SHM.

- Generally, if $E = \frac{1}{2}\alpha\dot{x}^2 + \frac{1}{2}\beta x^2$

  then $\dot{E} = \alpha\dot{x}\ddot{x} + \beta x\dot{x} = 0 \implies \ddot{x} + \frac{\beta}{\alpha}x = 0$

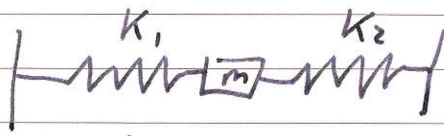
e.g mass on spring with gravity

- In equilibrium, $Kx_0 = Mg$
- If the mass is displaced further: $-k(x_0+x_1) + Mg = M\ddot{x}_1$

$$\implies \ddot{x}_1 + \frac{K}{m}x_1 = 0$$

- i.e gravity is irrelevant and system oscillates around equilibrium.

e.g mass on two springs

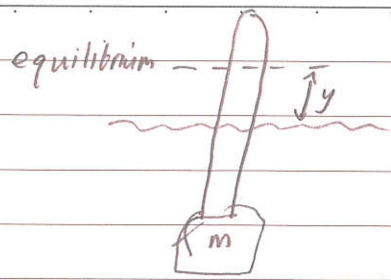


- For a small displacement:

$$-k_1 x - k_2 x = m\ddot{x} \implies \ddot{x} + \frac{k_1+k_2}{m}x = 0$$

i.e SHM with $\omega_0^2 = \frac{k_1+k_2}{m}$.

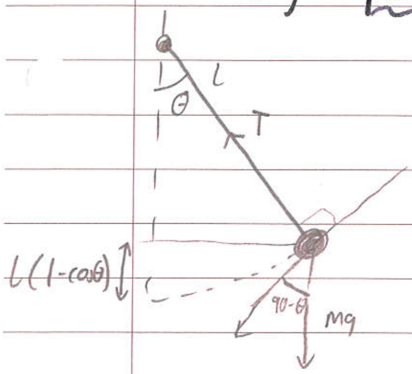
e.g <u>hydrometer</u>



- Archimedes' principle: buoyancy force is equal to the weight of the displaced fluid.
- When displaced from eq., the submerged volume changes by Ay (A = cross section area)

$\therefore\ m\ddot{y} = -\rho g A y \Rightarrow \ddot{y} + \frac{\rho g A}{m} y = 0.$

e.g <u>pendulum</u>



- Resolving perp. to tension, we have $ml\ddot{\theta} = -mg\sin\theta$.
- For small angular displacements:

$\sin\theta \simeq \theta \Rightarrow ml\ddot{\theta} = -mg\theta$

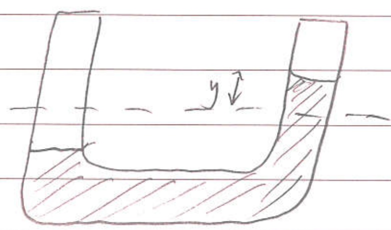
$\therefore\ \ddot{\theta} + \frac{g}{l}\theta = 0$.

- Alternatively, we can argue by energy:

$PE = mgl(1-\cos\theta) \simeq \frac{1}{2}mgl\theta^2$ for $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$.

- Then we use the energy formula with $\alpha = ml^2$, $\beta = mgl$

e.g <u>water in U-tube</u>

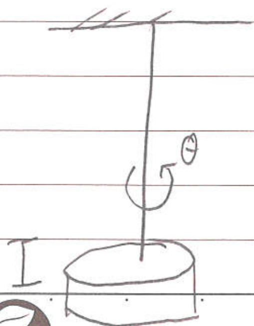


$PE = (\rho A y) g y = \rho A g y^2$

$KE = \frac{1}{2}\rho A l \dot{y}^2$

$\alpha = \rho A l,\ \beta = 2\rho A g \Rightarrow \omega_0^2 = 2g/l$

e.g <u>torsional oscillator</u>

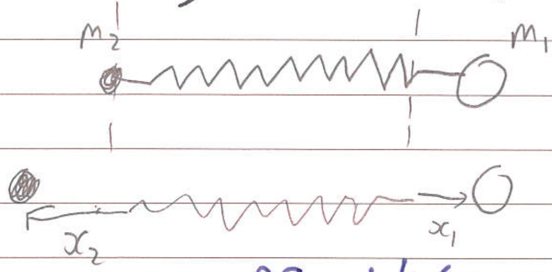


$PE = \frac{1}{2}\tau\theta^2$ ← τ is the torsional stiffness, Nm rad^{-1}.

$KE = \frac{1}{2}I\dot{\theta}^2$ with $I = \frac{1}{2}mR^2 \Rightarrow KE = \frac{1}{4}mR^2\dot{\theta}^2$

Then $\omega_0^2 = \frac{\tau}{I}$

e.g mass-spring-mass



- For a displacement with no net external force, the centre of mass can't move
- So we know $m_1 x_1 = m_2 x_2$

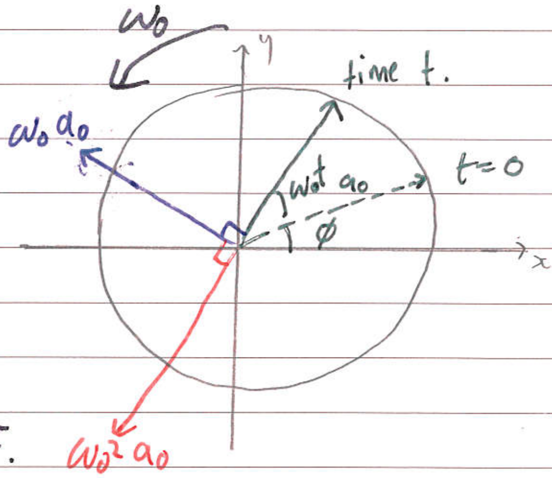
$$PE = \frac{1}{2} k (x_1 + x_2)^2 = \frac{1}{2} k x_1^2 \left(1 + \frac{m_1}{m_2}\right)^2$$

$$KE = \frac{1}{2} m_1 \left(1 + \frac{m_1}{m_2}\right) \dot{x}_1^2$$

$$\therefore \text{ SHM with } \omega_0^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$

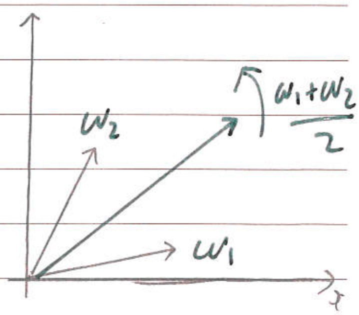
- This defines the reduced mass: $\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1}$

Phasor diagrams



- SHM can be visualised as rotation around a circle of radius a_0.
- x, $\dot{x}$ and $\ddot{x}$ all rotate at ω_0.
- We can use this to analyse the superposition of SHMs
  - ↳ must also be SHM because linear ODE.

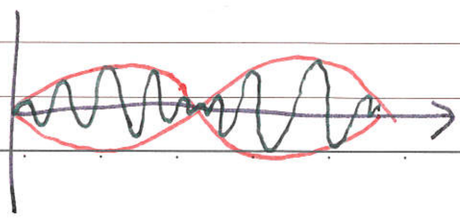
e.g superposition of ~~b~~ frequencies to form beats



- The phase difference is $(\omega_2 - \omega_1)t$, so the frequency is $\omega_2 - \omega_1$.
- Alternatively:

$$x = a_0(\cos\omega_1 t + \cos\omega_2 t)$$

$$\equiv 2a_0 \underbrace{\cos\left(\frac{\omega_2+\omega_1}{2} t\right)}_{\text{Fast}} \underbrace{\cos\left(\frac{\omega_2-\omega_1}{2} t\right)}_{\text{Slow}}$$



Complex representation of SHM

- We can derive from the phasor diagram: $z = a_0 e^{i(\omega_0 t + \phi)} = A e^{i\omega_0 t}$
  - ↳ the real part of z undergoes SHM.
  - ↳ all of the previous formulae follow.

- This expression satisfies $\ddot{z} + \omega_0^2 z = 0$.
- The total energy can be written as: $E = \frac{1}{2} k |z|^2$

Damped Harmonic Motion

- Damping is modelled as a resistive force proportional to velocity
  i.e $m\ddot{x} = -kx - b\dot{x}$ for a spring.
- The general form of the damped harmonic motion eq is:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

- For a spring, $\gamma = b/2m$ and $\omega_0^2 = k/m$ as before.
- Both γ and ω_0 have dimensions $1/\text{time}$, but with diff interpretations:
  - $T = 2\pi/\omega_0$ is the period
  - $\tau = \frac{1}{2\gamma}$ is a decay time

- Substituting $x = Ae^{-pt}$, the most general solution is:

$$x = Ae^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t} + Be^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t}$$

  ↳ the two constants fix the initial position and velocity.

Heavy damping $\gamma > \omega_0$

p is real ($p \equiv \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$), so the solution is the sum of two exponentials.

Light damping $\gamma < \omega_0$

Because $\gamma < \omega_0$, $p = \gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = \gamma \pm i\omega_d$

$$\therefore \quad z = Ae^{-\gamma t} e^{i\sqrt{\omega_0^2-\gamma^2}\, t}$$

with $A \equiv a_0 e^{i\phi}$

$$\therefore x(t) = Re(z) = a_0 e^{-\gamma t} \cos(\omega_d t + \phi)$$

The frequency of the oscillations does not change even though amplitude diminishes.

Critical damping $\gamma = \omega_0$

- Different solution in this case: $x = Ae^{-\gamma t} + Bt\, e^{-\gamma t}$
- Fastest decaying system: no osc, and minimal friction.

Comparing oscillators

- The logarithmic decrement measures how much the amplitude of a lightly damped oscillator drops per cycle

$$\frac{a_{n+1}}{a_n} = \frac{e^{-\gamma t_{n+1}}}{e^{-\gamma t_n}} = e^{-\gamma T}$$

$$\Rightarrow \quad \Delta = \frac{2\pi\gamma}{\omega_d}.$$

↳ a good oscillator has small Δ.

- Alternatively, the Quality factor Q of an oscillator is the number of radians of osc for energy to fall by a factor of e.

$$Q = \frac{\omega_0}{2\gamma}.$$

$\omega_d \simeq \omega_0$ for good oscillators, so $\Delta \approx \frac{\pi}{Q}$

Forced Oscillations

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\cos\omega t.$$

- For low-freq responce, we can ignore velocity and acc

↳ $x = \frac{f}{\omega_0^2}\cos\omega t$

- For high freq response, we can ignore velocity and displacement

↳ $x = -\frac{f\cos\omega t}{\omega^2}$

- At resonance, $\ddot{x} + \omega_0^2 x = 0$

↳ $x = \frac{f\sin(\omega_0 t)}{2\gamma\omega_0}$

More generally: $\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = fe^{i\omega t}$

Solved by $z = Ae^{i\omega t}$ with $A = \frac{f}{\omega_0^2 - \omega^2 + 2i\gamma\omega} = a_0 e^{i\phi}$.

a_0 and ϕ can be found using the standard methods.

Power and resonance

$$P_{av} = \langle Fv\rangle = \langle b\dot{x}^2\rangle = \frac{1}{2} b \frac{f^2}{((\omega_0^2-\omega^2)/\omega)^2 + 4\gamma^2}$$

From this we can derive the width at half power $\delta\omega$.

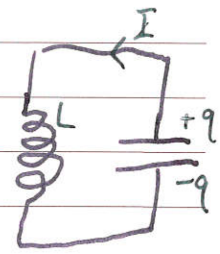
$$\omega_{hp} = \mp\gamma + \sqrt{\omega_0^2 + \gamma^2} \Rightarrow \delta\omega = 2\gamma$$

In terms of the quality factor:

$$\frac{\delta\omega}{\omega_0} = \frac{1}{Q}$$

i.e high quality oscillators have a very narrow resonance peak.

Electrical Oscillations



Consider a charged cap in a circuit with an inductor
- Current starts to flow, reducing q and thus reducing the voltage drop over both circuit elements
- Thus $\dot{I}$ ~~is negative, so current~~ decreases but is still positive, so $I\uparrow$ and the cap discharges faster.
- When $q=0$, the cap begins to charge negatively, causing the current to decrease.
- Thus, the system oscillates.

By kirchhoff's voltage law: $-L\dot{I} - \frac{q}{C} = 0$

$$\Rightarrow \quad \ddot{q} + \frac{1}{LC} q = 0$$

i.e SHM with $\omega_0^2 = 1/LC$.

↳ cap provides 'restoring force'

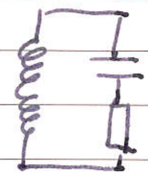
↳ inductor provides inertia

Alternatively, via cons energy:

$$E = \frac{1}{2C} q^2 + \frac{1}{2} L \dot{q}^2 \Rightarrow \text{SHM}$$

RLC circuits

A resistor acts as a damper because it dissipates energy



$$\dot{E} = \frac{1}{C} q\dot{q} + L\dot{q}\ddot{q} = -\dot{q}^2 R \Rightarrow \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC} q = 0$$

- This gives different solutions depending on the damping regime as before. For light damping:

$$q = q_0 e^{-\frac{R}{2L}t} \cos(\omega_d t + \phi), \text{ with } Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Alternating current

- An AC power source produces voltage $V = V_0 \cos\omega t$
- The rms voltage is given by $V_{rms} = \sqrt{\langle V^2 \rangle}$

Resistor:
- $I(t) = \frac{V_0}{R}\cos\omega t$ $\quad P = IV = \frac{V_0^2}{R}\cos\omega t$

Capacitor
- $Q = CV = CV_0\cos\omega t \Rightarrow I(t) = \omega C V_0 \cos(\omega t + \pi/2)$
- Current oscillates $\frac{1}{4}$ cycle ahead of voltage
- $P = VI = -\frac{1}{2}\omega C V_0^2 \sin(2\omega t)$

Inductor
- $\dot{I} = \frac{V_0\cos\omega t}{L} \Rightarrow I(t) = \frac{V_0}{L\omega}\cos(\omega t - \pi/2)$
- I maxes when sign of voltage changes, i.e $\frac{1}{4}$ cycle behind V.
- $P = \frac{1}{2}\frac{V_0^2}{L\omega}\sin(2\omega t)$.

- We can treat osc current/voltage/charge as the real parts of complex quantities: $V = V_0 e^{i\omega t}$, $I = I_0 e^{i\omega t}$ with V_0, I_0 complex
- We define impedance as the complex generalisation of resistance:

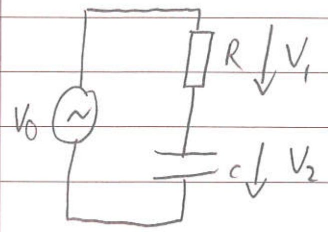
$$Z = \frac{V_0}{I_0}, \quad \text{with } |Z| = \frac{V_0}{I_0} \text{ and } \arg(Z) = -\phi.$$

- Resistor: $Z = \frac{V}{I} = \frac{V_0 e^{i\omega t}}{V_0 e^{i\omega t}/R} = R.$
  - Z is real because V and I in phase.
- Capacitor: $Z = \frac{V}{I} = \frac{V_0 e^{i\omega t}}{i\omega C V_0 e^{i\omega t}} = \frac{1}{i\omega C}$
- Inductor: $Z = \frac{V}{I} = \frac{V_0 e^{i\omega t}}{-i\frac{V_0}{L\omega}e^{i\omega t}} = i\omega L$

- Kirchhoff's laws apply to AC circuits with complex numbers

· Impedances add in series/parallel just like with DC circuits

e.g RC filter



Potential divider $\therefore \frac{V_2}{V_0} = \frac{1}{1+i\omega RC}$

with $\left|\frac{V_2}{V_0}\right| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$ and $\arg\left(\frac{V_2}{V_0}\right) = -\tan^{-1}(\omega RC)$

Thus the ~~vo~~ voltage across the cap can be used to 'remove' ~~low~~ high frequencies : low-pass filter.

Electrical resonance

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{V_0}{L}\cos\omega t$$

· Resonance when $\omega = 1/\sqrt{LC}$

↳ current ~~from~~ in circuit maximised