IA Oscillating Systems

No. 1
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- A system oscillates if $x(t) = x(t+T)$ for some Period T.
- The frequency of osc. is defined as $\nu = \frac{1}{T}$ (s^{-1})
- The angular freq is $\omega = 2\pi/T$
- For an object to undergo simple harmonic motion (SHM):
  - must be some inertia
  - and a restoring force $\propto$ (- displacement)

  $F = -kx = ma$ for a spring $\Longrightarrow$ $\ddot{x} = -\frac{k}{m}x$
- If we sub $x = a_0 \cos \omega t$ we find $\omega = \sqrt{\frac{k}{m}}$

- The general equation for undamped SHM is $\ddot{x} + \omega_0^2 x = 0$
- Solved by:

$$\left.\begin{aligned} x(t) &= a_0 \cos(\omega t + \phi) \\ x(t) &= A\cos(\omega_0 t) + B\sin(\omega_0 t) \end{aligned}\right\} \text{ 2 unknowns}$$

- SHM is unique among oscs because freq is independent to amplitude.
- The velocity and acceleration can be found by differentiating:

$$\dot{x}(t) = -\omega_0 a_0 \sin(\omega_0 t + \phi) \quad \left(\tfrac{1}{4} \text{ cycle ahead}\right)$$
$$\ddot{x}(t) = -\omega_0^2 a_0 \cos(\omega_0 t + \phi). \quad \left(\tfrac{1}{2} \text{ cycle ahead}\right).$$

we then see that $v_{max} = -\omega_0 a_0$ and $a_{max} = -\omega_0^2 a_0$

- We can instead analyse systems in terms of energy. For a spring:

$$KE = \frac{1}{2}m\dot{x}^2 \qquad PE = \int_0^x kX\,dX = \frac{1}{2}kx^2$$

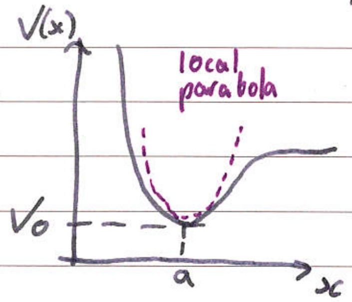
$$\Longrightarrow \quad \begin{aligned} KE &= \tfrac{1}{2} k a_0^2 \sin^2(\omega_0 t + \phi) \\ PE &= \tfrac{1}{2} k\, a_0^2 \cos^2(\omega_0 t + \phi) \\ E_{total} &= \tfrac{1}{2} k a_0^2 \end{aligned}$$

- Thus the energy terms oscillate twice as fast (e.g. write $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$)
- Because $\langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle = 1/2$, we know that $\langle KE \rangle = \langle PE \rangle = \frac{1}{4}ka_0^2$

- We can instead derive SHM using the conservation of energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \Longrightarrow \dot{E} = m\dot{x}\ddot{x} + kx\dot{x} \Rightarrow \ddot{x} + \frac{k}{m}x = 0.$$

- Thus, any time a particle moves in a quadratic PE, it will undergo SHM.
- This can be applied to more complex potentials:



Taylor expanding about $x=a$:

$$V(a+\Delta x) = V(a) + V'(a)\Delta x + \frac{V''(a)}{2}(\Delta x)^2 + \dots$$

At the minimum, $V'(a)=0$

$$\therefore \quad V(a+\Delta x) \approx V(a) + \frac{1}{2}V''(a)(\Delta x)^2$$

- Small peturbations can be approxed by quadratic potentials
  ↳ modeled with SHM.

- Generally, if $E = \frac{1}{2}\alpha\dot{x}^2 + \frac{1}{2}\beta x^2$

  then $\dot{E} = \alpha\dot{x}\ddot{x} + \beta x\dot{x} = 0 \implies \ddot{x} + \frac{\beta}{\alpha}x = 0$

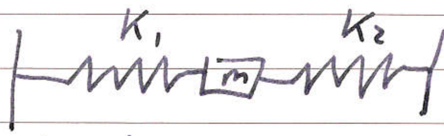
e.g <u>mass on spring with gravity</u>

- In equilibrium, $Kx_0 = Mg$
- If the mass is displaced further: $-k(x_0+x_1) + Mg = M\ddot{x}_1$

$$\implies \ddot{x}_1 + \frac{k}{m}x_1 = 0$$

- i.e gravity is irrelevant and system oscillates around equilibrium.

e.g <u>mass on two springs</u>

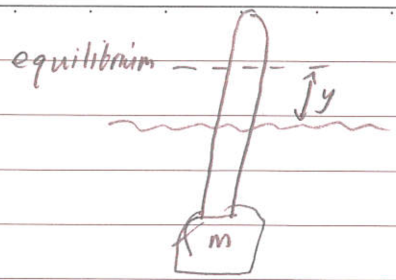


- For a small displacement:

$$-k_1 x - k_2 x = m\ddot{x} \implies \ddot{x} + \frac{k_1+k_2}{m}x = 0$$

i.e SHM with $\omega_0^2 = \frac{k_1+k_2}{m}$.

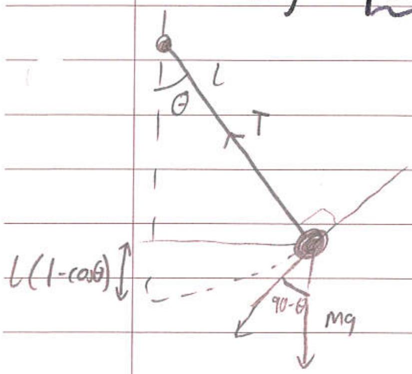
e.g **hydrometer**



- Archimedes' principle: buoyancy force is equal to the weight of the displaced fluid.
- When displaced from eq., the submerged volume changes by Ay (A = cross section area)

$$\therefore \; m\ddot{y} = -\rho g A y \Rightarrow \ddot{y} + \frac{\rho g A}{m} y = 0.$$

e.g **pendulum**



- Resolving perp. to tension, we have $ml\ddot{\theta} = -mg\sin\theta$.
- For small angular displacements:

$$\sin\theta \simeq \theta \Rightarrow ml\ddot{\theta} = -mg\theta$$

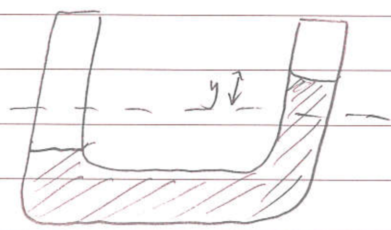
$$\therefore \; \ddot{\theta} + \frac{g}{l}\theta = 0.$$

- Alternatively, we can argue by energy:

$$PE = mgl(1-\cos\theta) \simeq \frac{1}{2}mgl\theta^2 \quad \text{for } \cos\theta \simeq 1 - \frac{1}{2}\theta^2.$$

- Then we use the energy formula with $\alpha = ml^2$, $\beta = mgl$

e.g **water in U-tube**

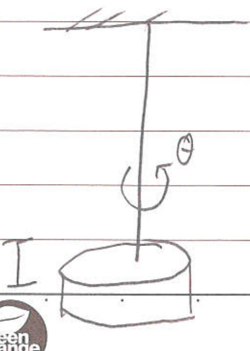


$$PE = (\rho A y)gy = \rho A g y^2$$

$$KE = \frac{1}{2}\rho A l \dot{y}^2$$

$$\alpha = \rho A l, \quad \beta = 2\rho A g \Rightarrow \omega_0^2 = 2g/l$$

e.g **torsional oscillator**

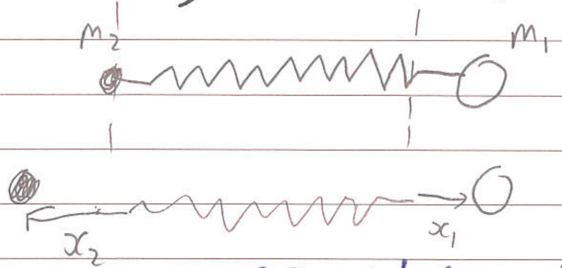


$$PE = \frac{1}{2}\tau\theta^2 \quad \leftarrow \tau \text{ is the torsional stiffness, } \mathrm{Nm\ rad^{-1}}.$$

$$KE = \frac{1}{2}I\dot{\theta}^2 \text{ with } I = \frac{1}{2}mR^2 \Rightarrow KE = \frac{1}{4}mR^2\dot{\theta}^2$$

Then $\omega_0^2 = \dfrac{\tau}{I}$

e.g mass-spring-mass



- For a displacement with no net external force, the centre of mass can't move
- So we know $m_1 x_1 = m_2 x_2$

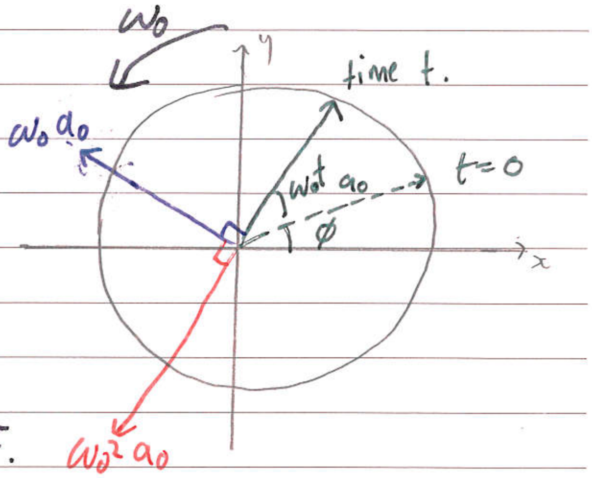
$$PE = \frac{1}{2} k (x_1 + x_2)^2 = \frac{1}{2} k x_1^2 \left(1 + \frac{m_1}{m_2}\right)^2$$

$$KE = \frac{1}{2} m_1 \left(1 + \frac{m_1}{m_2}\right) \dot{x}_1^2$$

$\therefore$ SHM with $\omega_0^2 = k\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$

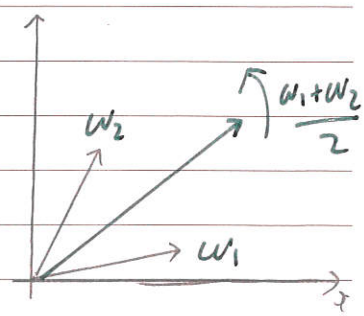
- This defines the reduced mass : $\mu = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1}$

Phasor diagrams



- SHM can be visualised as rotation around a circle of radius a_0.
- x, $\dot{x}$ and $\ddot{x}$ all rotate at ω_0.
- We can use this to analyse the superposition of SHMs
  - ↳ must also be SHM because linear ODE.

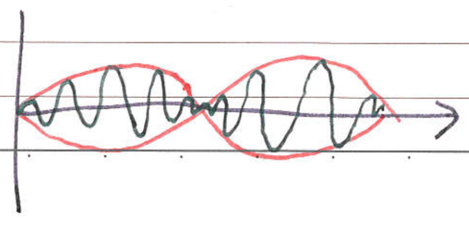
e.g superposition of ~~ta~~ frequencies to form beats



- The phase difference is $(\omega_2 - \omega_1)t$, so the frequency is $\omega_2 - \omega_1$.
- Alternatively:

$$x = a_0 (\cos\omega_1 t + \cos\omega_2 t)$$

$$\equiv 2a_0 \underbrace{\cos\left(\frac{\omega_2+\omega_1}{2} t\right)}_{\text{Fast}} \underbrace{\cos\left(\frac{\omega_2-\omega_1}{2} t\right)}_{\text{Slow}}$$



Complex representation of SHM

- We can derive from the phasor diagram: $z = a_0 e^{i(\omega_0 t + \phi)} = Ae^{i\omega_0 t}$
  - ↳ the real part of z undergoes SHM.
  - ↳ all of the previous formulae follow.

- This expression satisfies $\ddot{z} + \omega_0^2 z = 0$.
- The total energy can be written as: $E = \frac{1}{2} k |z|^2$

Damped Harmonic Motion

- Damping is modelled as a resistive force proportional to velocity
  i.e $m\ddot{x} = -kx - b\dot{x}$ for a spring.
- The general form of the damped harmonic motion eq is:

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$$

- For a spring, $\gamma = b/2m$ and $\omega_0^2 = k/m$ as before.
- Both γ and ω_0 have dimensions $1/\text{time}$, but with diff interpretations:
  - $T = 2\pi/\omega_0$ is the period
  - $\tau = \frac{1}{2\gamma}$ is a decay time

- Substituting $x = Ae^{-pt}$, the most general solution is:

$$x = Ae^{(-\gamma - \sqrt{\gamma^2 - \omega_0^2})t} + Be^{(-\gamma + \sqrt{\gamma^2 - \omega_0^2})t}$$

  ↳ the two constants fix the initial position and velocity.

Heavy damping $\gamma > \omega_0$

p is real $(p \equiv \gamma \pm \sqrt{\gamma^2 - \omega_0^2})$, so the solution is the sum of two exponentials.

Light damping $\gamma < \omega_0$

Because $\gamma < \omega_0$, $p = \gamma \pm i\sqrt{\omega_0^2 - \gamma^2} = \gamma \pm i\omega_d$

$$\therefore \quad z = Ae^{-\gamma t} e^{i\sqrt{\omega_0^2-\gamma^2}\, t}$$

with $A \equiv a_0 e^{i\phi}$

$$\therefore x(t) = Re(z) = a_0 e^{-\gamma t} \cos(\omega_d t + \phi)$$

The frequency of the oscillations does not change even though amplitude diminishes.

Critical damping $\gamma = \omega_0$

- Different solution in this case: $x = Ae^{-\gamma t} + Bt\, e^{-\gamma t}$
- Fastest decaying system: no osc, and minimal friction.

Comparing oscillators

- The logarithmic decrement measures how much the amplitude of a lightly damped oscillator drops per cycle

$$\frac{a_{n+1}}{a_n} = \frac{e^{-\gamma t_{n+1}}}{e^{-\gamma t_n}} = e^{-\gamma T}$$

$$\Rightarrow \quad \Delta = \frac{2\pi\gamma}{\omega_d}.$$

↳ a good oscillator has small Δ.

- Alternatively, the Quality factor Q of an oscillator is the number of radians of osc for energy to fall by a factor of e.

$Q = \dfrac{\omega_0}{2\gamma}$. $\quad \omega_d \simeq \omega_0$ for good oscillators, so $\Delta \approx \dfrac{\pi}{Q}$

Forced Oscillations

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = f\cos\omega t.$$

- For low-freq responce, we can ignore velocity and acc

↳ $x = \frac{f}{\omega_0^2}\cos\omega t$

- For high freq response, we can ignore velocity and displacement

↳ $x = -\frac{f\cos\omega t}{\omega^2}$

- At resonance, $\ddot{x} + \omega_0^2 x = 0$

↳ $x = \frac{f\sin(\omega_0 t)}{2\gamma\omega_0}$

More generally: $\ddot{z} + 2\gamma\dot{z} + \omega_0^2 z = fe^{i\omega t}$

Solved by $z = Ae^{i\omega t}$ with $A = \frac{f}{\omega_0^2 - \omega^2 + 2i\gamma\omega} = a_0 e^{i\phi}$.

a_0 and ϕ can be found using the standard methods.

Power and resonance

$$P_{av} = \langle Fv\rangle = \langle b\dot{x}^2\rangle = \frac{1}{2}b\frac{f^2}{((\omega_0^2-\omega^2)/\omega)^2 + 4\gamma^2}$$

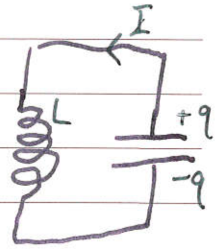
From this we can derive the width at half power $\delta\omega$.

$$\omega_{hp} = \mp\gamma + \sqrt{\omega_0^2 + \gamma^2} \Rightarrow \delta\omega = 2\gamma$$

In terms of the quality factor:

$\frac{\delta\omega}{\omega_0} = \frac{1}{Q}$ i.e high quality oscillators have a very narrow resonance peak.

Electrical Oscillations



Consider a charged cap in a circuit with an inductor

- Current starts to flow, reducing q and thus reducing the voltage drop over both circuit elements
- Thus $\dot{I}$ ~~is negative, so current~~ decreases but is still positive, so $I\uparrow$ and the cap discharges faster.
- When $q=0$, the cap begins to charge negatively, causing the current to decrease.
- Thus, the system oscillates.

By Kirchhoff's Voltage law: $-L\dot{I} - \frac{q}{C} = 0$

$$\Rightarrow \quad \ddot{q} + \frac{1}{LC}q = 0$$

i.e SHM with $\omega_0^2 = 1/LC$.

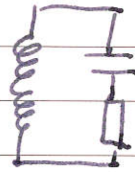
↳ cap provides 'restoring force'

↳ inductor provides inertia

Alternatively, via cons energy:

$$E = \frac{1}{2C}q^2 + \frac{1}{2}L\dot{q}^2 \Rightarrow \text{SHM}$$

RLC circuits



A resistor acts as a damper because it dissipates energy

$$\dot{E} = \frac{1}{C}q\dot{q} + L\dot{q}\ddot{q} = -\dot{q}^2 R \Rightarrow \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0$$

- This gives different solutions depending on the damping regime as before. For light damping:

$$q = q_0 e^{-\frac{R}{2L}t}\cos(\omega_d t + \phi), \text{ with } Q = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Alternating current

- An AC power source produces voltage $V = V_0 \cos\omega t$
- The rms voltage is given by $V_{rms} = \sqrt{\langle V^2 \rangle}$

Resistor:

- $I(t) = \frac{V_0}{R}\cos\omega t \qquad P = IV = \frac{V_0^2}{R}\cos\omega t$

Capacitor

- $Q = CV = CV_0\cos\omega t \Rightarrow I(t) = \omega C V_0 \cos(\omega t + \pi/2)$
- Current oscillates $\frac{1}{4}$ cycle ahead of voltage
- $P = VI = -\frac{1}{2}\omega C V_0^2 \sin(2\omega t)$

Inductor

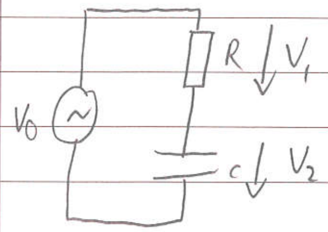
- $\dot{I} = \frac{V_0 \cos\omega t}{L} \Rightarrow I(t) = \frac{V_0}{L\omega}\cos(\omega t - \pi/2)$
- I maxes when sign of voltage changes, i.e. $\frac{1}{4}$ cycle behind V.
- $P = \frac{1}{2}\frac{V_0^2}{L\omega}\sin(2\omega t)$.

- We can treat osc current/voltage/charge as the real parts of complex quantities: $V = V_0 e^{i\omega t}$, $I = I_0 e^{i\omega t}$ with V_0, I_0 complex
- We define impedance as the complex generalisation of resistance:
  $Z = \frac{V_0}{I_0}$, with $|Z| = \frac{V_0}{I_0}$ and $\arg(Z) = -\phi$.
- Resistor: $Z = \frac{V}{I} = \frac{V_0 e^{i\omega t}}{V_0 e^{i\omega t}/R} = R$.
  - Z is real because V and I in phase.
- Capacitor: $Z = \frac{V}{I} = \frac{V_0 e^{i\omega t}}{i\omega C V_0 e^{i\omega t}} = \frac{1}{i\omega C}$
- Inductor: $Z = \frac{V}{I} = \frac{V_0 e^{i\omega t}}{-i\frac{V_0}{L\omega}e^{i\omega t}} = i\omega L$.

- Kirchhoff's laws apply to AC circuits with complex numbers

· Impedances add in series/parallel, just like with DC circuits

e.g <u>RC filter</u>



Potential divider $\therefore \frac{V_2}{V_0} = \frac{1}{1+i\omega RC}$

with $\left|\frac{V_2}{V_0}\right| = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$ and $\arg\left(\frac{V_2}{V_0}\right) = -\tan^{-1}(\omega RC)$

Thus the ~~vo~~ voltage across the cap can be used to 'remove' ~~low~~ high frequencies : low-pass filter.

Electrical resonance

$$\ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = \frac{V_0}{L}\cos\omega t$$

· Resonance when $\omega = 1/\sqrt{LC}$

↳ current ~~from~~ in circuit maximised