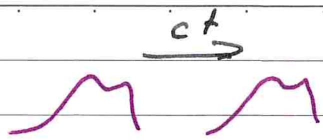
IA Waves



- for a travelling wave, $c = f\lambda$
- For a wave travelling to the right, its displacement at time t can be found by looking backwards by ct.
  ↳ thus the wave function has the form $\psi = f(x - ct)$
- Any function of this form will satisfy the wave equation:

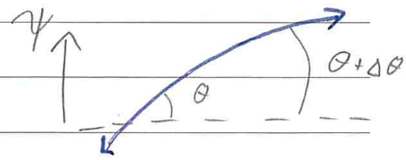
$$\frac{\partial^2 \psi}{\partial t^2} = c^2 \frac{\partial^2 \psi}{\partial x^2}$$

↳ linear in ψ so principle of superposition applies.
i.e waves that meet don't interact.
↳ likewise, a disturbance can be split into +ve and −ve components

$$\psi = f(x - ct) + g(x + ct)$$

e.g waves on a stretched string

- Consider a segment Δx with tension T and density ρ.
  ↳ $F_y = T\sin(\theta + \Delta\theta) - T\sin\theta = T\Delta\theta$
  ↳ by N II : $T\Delta\theta = \rho \Delta x \frac{\partial^2 \psi}{\partial t^2}$
  ↳ with $\frac{\partial \psi}{\partial x} \sim \theta$, $\frac{\partial^2 \psi}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 \psi}{\partial x^2}$



$\therefore$ wave motion with $c = \sqrt{\frac{T}{\rho}}$

- Both $A\cos(kx - \omega t)$ and $B\sin(kx - \omega t)$ are solutions:
  $\psi = \exp\{i(kx - \omega t)\}$ is also a solution
  $k = \frac{2\pi}{\lambda}$ is the wavenumber, and $\omega = \frac{2\pi}{T}$ is the angular frequency

- $c \neq$ particle speed, which is given by $\frac{\partial \psi}{\partial t}$

- The KE / unit length for a string is $\frac{1}{2}\rho\left(\frac{\partial \psi}{\partial t}\right)^2$
- The PE / unit length is $\frac{1}{2}T\left(\frac{\partial \psi}{\partial x}\right)^2$
  $\therefore$ total energy density is $\frac{1}{2}\rho\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial \psi}{\partial x}\right)^2$.

· From this we can calculate the time-avg energy density as

$$\frac{1}{2}\rho A^2\omega^2$$

↳ $\therefore \Delta E = \frac{1}{2}\rho A^2\omega^2\Delta x$. But $\Delta x = c\Delta t$:

$$\therefore P = \frac{1}{2}\rho A^2\omega^2 c$$

· In more dimensions, the wavenumber becomes a wavevector:

$$\Psi(\underset{\sim}{r}, t) = \mathrm{Re}\left\{ A e^{i(\omega t - \underset{\sim}{k}\cdot\underset{\sim}{r})} \right\}$$

where $\underset{\sim}{k} = (k_x, k_y, k_z)$.

↳ $\omega^2 = c^2k^2 \implies \frac{\partial^2\Psi}{\partial t^2} = c^2\nabla^2\Psi$

Standing waves

· Two sinusoidal waves travelling in opposite directions ~~may~~ will interfere to produce a standing wave:

$$\Psi(x,t) = A[\cos(\omega t - kx) + \cos(\omega t + kx)] = 2A\cos kx \cos\omega t$$

↳ each point oscillates but there is no overall transfer of energy

↳ all osc has the same freq and phase, but different amplitude

↳ there will be nodes and antinodes

· The exact wave will depend on boundary conditions, e.g because fixed ends reflect with a phase change of π.

- e.g for a string of length L fixed on both ends

$\Psi(0,t) = \Psi(L,t) = 0$, with general $\Psi(x,t) = (A\cos kx + B\sin kx)\cos\omega t$.

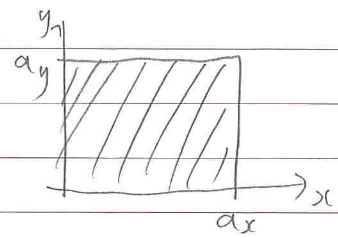
$\Rightarrow A = 0$ or $k = \frac{n\pi}{L}$, $n = 1, 2, 3$.

↳ i.e boundary conditions lead to quantised wavenumbers

$\therefore f = cn/2L \leftarrow n=1$ gives the fundamental frequency, $n>1$ gives harmonics

- In pipes, we must distinguish between particle displacement Ψ and atmospheric pressure, $\propto \frac{\partial \Psi}{\partial x}$.
  - ↳ $\Psi(x,t) = (A\cos kx + B\sin kx)\cos\omega t$ as before, but the B.C.s are now in terms of $\frac{\partial \Psi}{\partial x}$.
  - ↳ for an open end, $\partial \Psi / \partial x = 0$, i.e. pressure node (but Ψ antinode).
  - ↳ if one end is closed: $k = \frac{(2n-1)\pi}{2L} \Rightarrow f = \frac{c(2n-1)}{4L}$

- In 2D, we will also have a separated solution:
  $$\Psi(x,y,t) = X(x)\,Y(y)\cos\omega t$$
  with. $X(x) = A_x\cos(k_x x) + B_x\sin(k_x x)$
  $Y(y) = A_y\cos(k_y y) + B_y\sin(k_y y)$
- On a rectangular drumskin, this has the solution:
  $k_x = \frac{n\pi}{a_x}$, $k_y = \frac{n\pi}{a_y}$

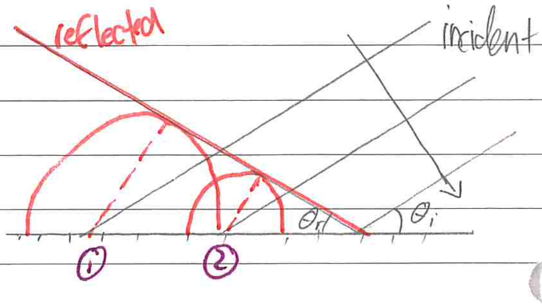


  $$\Rightarrow \frac{\omega^2}{c^2} = k_x^2 + k_y^2 = \frac{n^2\pi^2}{a_x^2} + \frac{m^2\pi^2}{a_y^2}$$
  and $\Psi(x,y,t) = B\sin\left(\frac{n\pi x}{a_x}\right)\sin\left(\frac{m\pi y}{a_y}\right)\cos\omega t$.

- This is easily extended to 3D. In a box:
  $\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2$ and $\Psi(x,y,z,t) \propto \sin k_x x \sin k_y y \sin k_z z \cos\omega t$

Optics

- **Huygens' Principle**: every point on a primary wavefront behaves as a source of secondary wavelets.
  - at a boundary, the first wavefront arrives and propagates
  - at a later time, the next wavefront arrives
  - connecting the cusps gives a reflected wave with $\theta_i = \theta_r$



- The **Law of Refraction** is: $\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$

  ↳ using the definition of the refractive index, we derive **Snell's law**

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

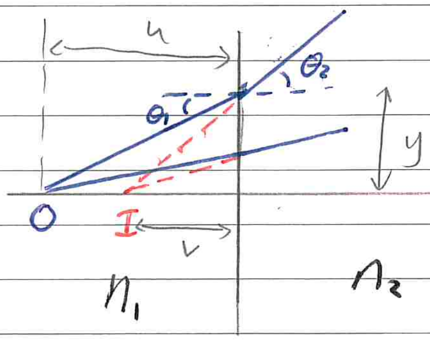
  ↳ if $n_1 > n_2$, there will be some values of θ_1 that make $\sin\theta_2 > 1$. Not possible, so there cannot be refraction. **Total internal reflection** instead.

- An object in a denser medium viewed from a less dense one will appear shorter
  - O is a distance u from the interface with $n_2 < n_1$
  - using a small angle approximation:

  ~~$\sin\theta_1$~~ $\tan\theta_1 = \frac{y}{u}$ $\quad \tan\theta_2 = \frac{y}{v}$

  ∴ and $\theta_1/\theta_2 \approx n_2/n_1$

$$\Rightarrow v = u\frac{n_2}{n_1}$$

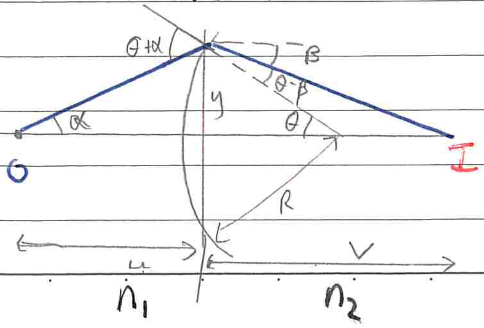


- To analyse spherical interfaces, we must make the **paraxial approximation**, ie rays are almost horizontal so $\sin\theta \sim \tan\theta \sim \theta$.

$n_1 \sin(\theta+\alpha) = n_2 \sin(\theta-\beta) \Rightarrow n_1(\theta+\alpha) = n_2(\theta-\beta)$

$\therefore n_1 \frac{y}{R} + n_1 \frac{y}{u} = n_2 \frac{y}{R} - n_2 \frac{y}{v}$

$\Rightarrow \frac{n_1}{u} + \frac{n_2}{v} = \frac{n_2 - n_1}{R}$ ← independent of α, all ~~pain~~ rays focused.



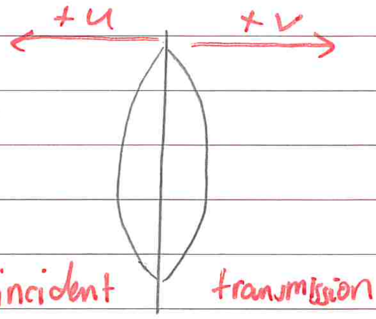
Lenses

- If we combine two curved interfaces, we obtain a lens.
- Using the thin lens approximation (i.e not much bending in the lens), we can arrive at the Lens-Maker's equation:

$$\frac{1}{u} + \frac{1}{v} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$ ← n_2 inside lens ← R_1 for left surface.

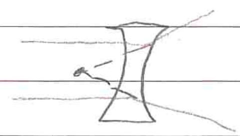
$\therefore$ for $u \to \infty$, $\boxed{\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$

- This implies that $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$

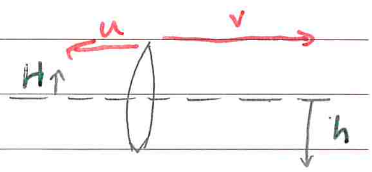


- Important sign conventions:
  - R_1 positive if incident convex (centre of curvature on right)
  - R_2 positive if centre of curvature on left.

- A converging lens has positive focal length.
- A diverging lens has negative focal length so brings parallel beams to a virtual focus.

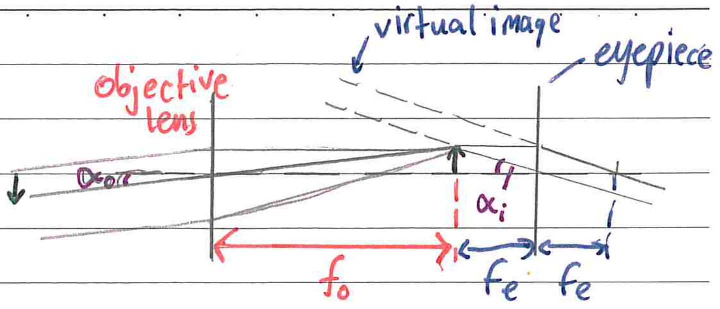


- Ray diagrams can be constructed using basic principles
  - parallel beam will pass through focus
  - beam through focus emerges parallel.
  - ray through centre of thing lens is unrefracted.
- The magnification of an object can be found using similar triangles: $M = \frac{h}{H} = \frac{v}{u}$



- The power of a lens is defined as $P \equiv \frac{1}{f}$

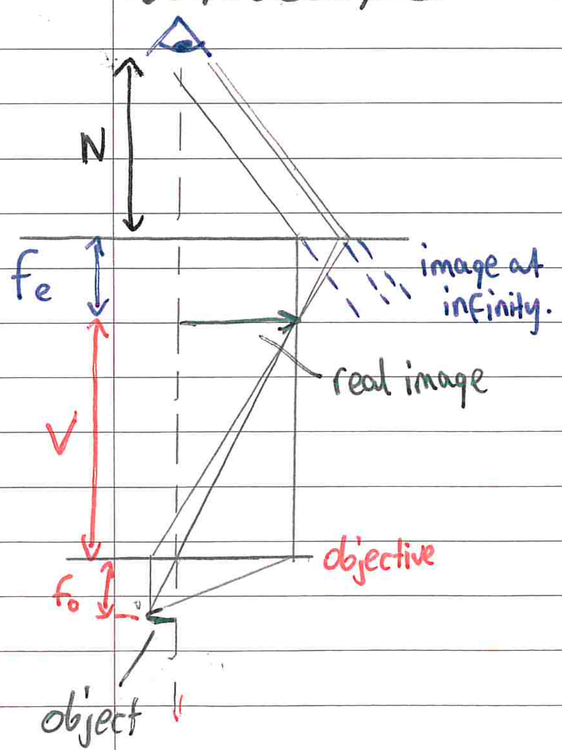
Astronomical telescopes



- For astronomical objects, $u \approx \infty$ so beams are near-parallel.
- The objective lens brings them to a focus at f_o, forming an inverted real image.
- If this happens to be located at the focal length of the eyepiece f_e, it will form a virtual image at infinity.
- Thus the telescope needs to be constructed such that the distance between lenses is the sum of their focal lengths.
- The magnification is then: $|M| = \dfrac{\alpha_i}{\alpha_o} = \dfrac{f_o}{f_e}$

↳ hence old telescopes are very long to maximise ~~α_o~~ f_o.

Compound light microscope



- The object is placed close to the focal length to produce a large real image

$$M_o = \frac{v}{u} \approx \frac{v}{f_o}$$

- This image is at the focal length of the eyepiece so produces // rays and a virtual image at infinity.

↳ distance from eye to eyepiece is the nearpoint distance $N \sim 25\,\text{cm}$

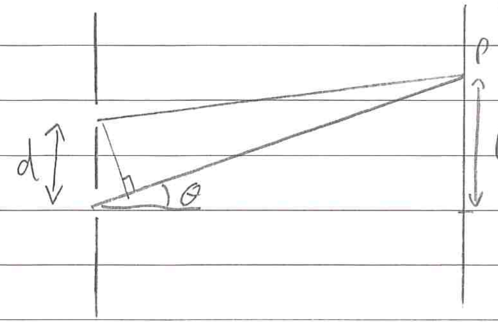
$$M_e = -\frac{N}{f_e}$$

- Combined:

$$M = M_o M_e = -\frac{v}{f_o}\frac{N}{f_e} \quad \text{(upright)}.$$

Interference & Diffraction

- Consider the interference pattern from twin slits on a distant screen.



- $r_2 = r_1 + d\sin\theta$

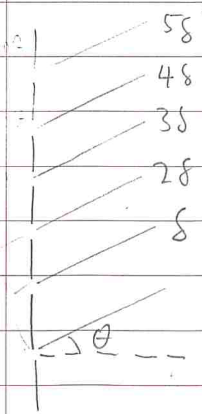
$$\Rightarrow \psi_p = A\cos(\omega t - kr_1) + A\cos(\omega t - kr_2)$$
$$= 2A\cos\left(\omega t - kr_1 + \tfrac{kd}{2}\sin\theta\right)\cos\left(\tfrac{kd}{2}\sin\theta\right)$$

- We can average out the time variation so that:

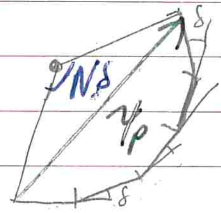
$$I_p \propto \psi_p^2 \propto \cos^2\left(\frac{kd}{2}\sin\theta\right)$$

- Thus there are maxima when $d\sin\theta = n\lambda$, i.e constructive interference because integer number of wavelengths.

- To analyse many slits, it is better to use phasors.



  - each wave is at a constant phase difference to the previous one, $\delta = kd\sin\theta$.
  - on a phasor diagram, this will be a section of a polygon.
  - maxima when the phasors add in phase $\Rightarrow \delta = \pm 2n\pi \Rightarrow d\sin\theta = n\lambda$
  - $\psi \propto N^2$
  - zero when $N\delta = 2\pi \Rightarrow \sin\theta = \frac{\lambda}{Nd}$
    ↳ i.e peak width $\propto \frac{1}{N}$.

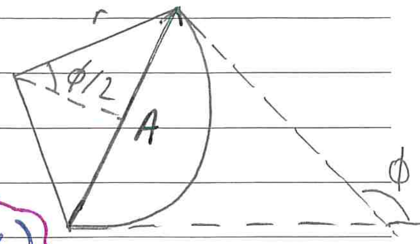


- By the Rayleigh criterion, two lines will be resolved if the maximum of one lies over the first zero of the other.

i.e $\Delta\theta = \dfrac{\lambda}{Nd}$

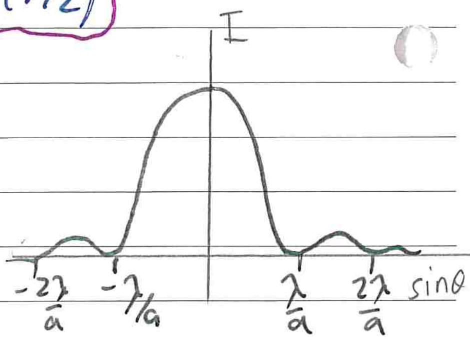
Slits of finite width

- Because of Huygens' principle, a finite slit can be treated as a grating with $N \to \infty$ slits. Thus the polygon becomes a circular arc
  - ↳ if the slit has width a, the total phase diff between the top and bottom is $\phi = ka\sin\theta$
  - ↳ by geometry, $A = 2r\sin\left(\frac{\phi}{2}\right)$.
  - ↳ the arc length is $NA_0 \equiv A_{max}$

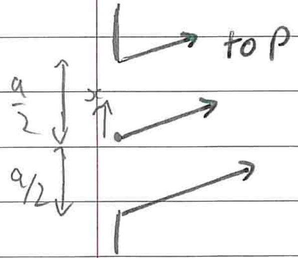


$$\therefore A = \frac{2A_{max}}{\phi}\sin\left(\frac{\phi}{2}\right) \Rightarrow A = A_{max}\,\text{sinc}(\phi/2)$$

  - ↳ thus there are minima when $\phi = 2n\pi$, or equivalently $\sin\theta \sim \theta \approx \frac{n\lambda}{a}$



- Alternatively, we can derive this by integration, considering phase diff relative to the centre



  - The phase diff at a height x will be $kx\sin\theta$.
  - Let $A_0 \equiv$ amplitude per unit length of slit
  - Hence the total amplitude at P, A_P, is:

$$A_P = A_0\int_{-a/2}^{a/2} \exp(ikx\sin\theta)\,dx$$

$$= \frac{2A_0}{k\sin\theta}\sin\left(k\left(\frac{a}{2}\right)\sin\theta\right).$$

- But $\phi = ka\sin\theta$, and $A_{max} = A_0 a$ if they all add in phase

$$\therefore A_P = A_{max}\,\text{sinc}(\phi/2) \text{ as before.}$$

Quantum physics

- The probability of finding a particle at a given position: $P = |\hat{\Psi}|^2$
  ↳ in QM we need to use the complex wavefunction.
- For a photon, $E = pc = hf$ (experimentally), allowing us to derive the de Broglie wavelength $\lambda = h/p$.
- The wavefunction for a particle moving in $+x$ is

$$\hat{\Psi}(x,t) = \Psi_0 \exp i(kx - \omega t)$$ ← always $kx - \omega t$ in QM.

↳ can be rewritten using $p = \hbar k$ and $E = \hbar\omega$ $\left(\hbar \equiv \frac{h}{2\pi}\right)$

$$\hat{\Psi}(x,t) = \Psi_0 \exp \frac{i}{\hbar}(px - Et).$$

The Schrödinger equation

- From $E = p^2/2m + V(x,t)$, we can derive:

$$E\hat{\Psi} = i\hbar \frac{\partial \hat{\Psi}}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \hat{\Psi}}{\partial x^2} + V(x,t)\hat{\Psi}$$

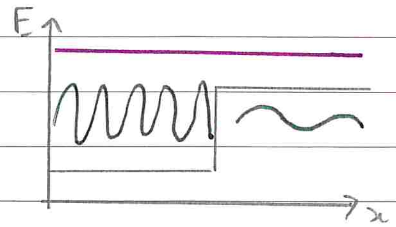
- If $V(x)$ is not time-varying and E is constant, we can derive the time-independent Schrödinger equation:

$$\frac{\partial^2 \hat{\Psi}}{\partial x^2} + \frac{2m}{\hbar^2}[E - V(x)]\hat{\Psi} = 0$$

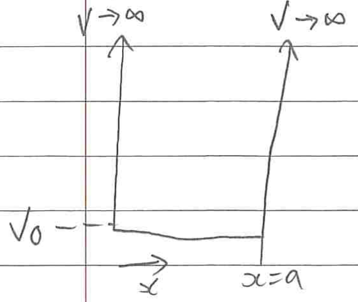
- Substituting the wave function of a single particle:

$$k = \frac{\sqrt{2m(E-V)}}{\hbar}$$

- Thus k varies based on V:



Potential wells (infinite)



- Outside the well, $\hat{\Psi} = 0$ so the particle cannot exist.
- Inside the well, $\hat{\Psi}(x,t) = \Psi_0 \exp i(kx - \omega t)$
  $\Rightarrow k^2 = 2m(E - V_0)/\hbar^2$
- Because $\pm k$ are solutions, the general solution is:

$$\hat{\Psi}(x,t) = A\exp i(kx - \omega t) + B\exp i(-kx - \omega t)$$

- $\hat{\Psi}(0,t) = 0 \Rightarrow B = -A \;\therefore\; \hat{\Psi} = C\sin kx \exp(-i\omega t)$
  - ↳ note that imaginary part disappears for $|\hat{\Psi}|^2 = \hat{\Psi}\hat{\Psi}^*$
- To satisfy $\Psi(a,t) = 0$, the wavenumber becomes quantised: $k_n = \frac{n\pi}{a}$
  - ↳ hence energy levels are quantised

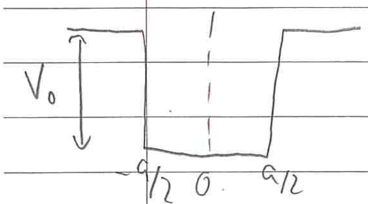
$$E_n - V_0 = \frac{\pi^2\hbar^2}{2ma^2} n^2$$

n is the quantum number

  - ↳ i.e heavy particles are easier to confine
  - ↳ more energy ~~req~~ for particles in small box ← uncertainty princ.
- The constant can be found by normalising:

$$\int_{-\infty}^{\infty} |\hat{\Psi}|^2 dx = 1 \quad \Rightarrow \quad \Psi = \sqrt{\frac{2}{a}} \sin\frac{\pi x}{a}\, e^{-i\omega t}$$

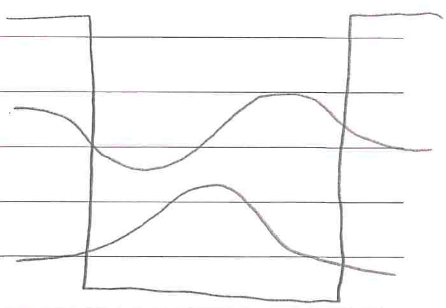
Finite potential well



- Inside the well, $E - V > 0 \Rightarrow$ oscillatory solution
- Outside, $E < V_0 \;\therefore\; k$ is imaginary.
  $k \equiv i\kappa \Rightarrow \Psi = Ae^{\pm\kappa x}e^{-i\omega t}$
- Hence inside:
  $\hat{\Psi}(x) = A\exp(ik_1 x) + B\exp(-ik_1 x)$, $\quad k_1 = \sqrt{2mE}/\hbar$
- Outside: $\hat{\Psi} = Ce^{-\kappa x} + De^{\kappa x}$

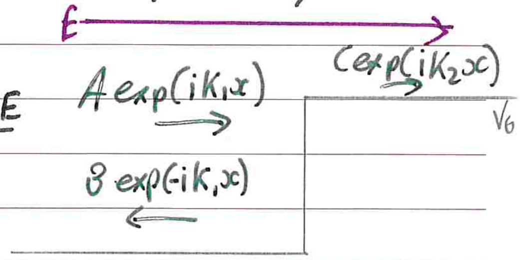
- Boundary conditions:
  - $\hat{\Psi}(x \to \pm\infty) = 0$
  - $\hat{\Psi}$ continuous ~~at b~~ everywhere, otherwise $\frac{\partial\Psi}{\partial x}$ (momentum) infinite.
  - $\frac{\partial\Psi}{\partial x}$ continuous everywhere, else $\frac{\partial^2\Psi}{\partial x^2}$ (energy) infinite.

· Outside the well, ψ decays exponentially
↳ evanescent wave



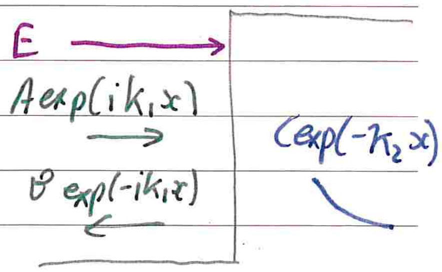
Step barriers

· Consider a constant beam of electrons with energy E travelling in $+x$ towards a potential step V_0, $E > V_0$.
· Classically, we wouldn't expect reflection. But in quantum, there is a finite amplitude reflection coefficient.



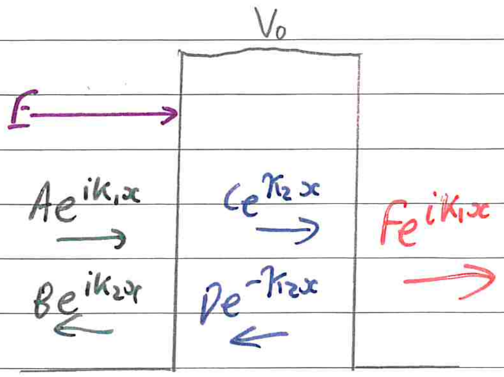
- to the left of the barrier:
  $$\psi = A\exp(ik_1x) + B\exp(-ik_1x) \quad \text{with} \quad k_1 = \frac{\sqrt{2mE}}{\hbar}$$
- to the right, there is no reflected wave so $\hat{\psi} = C\exp ik_2x$, $k_2 = \frac{\sqrt{2m(E-V)}}{\hbar}$
- we can then apply B.Cs to find $\frac{C}{A}$ and $\frac{B}{A}$, because we only care about the reflected proportion.

· For a step barrier with $E < V_0$, the wave is totally reflected:



- to the right, there is a decaying evanescent wave. No $\exp(+\kappa_2x)$ because unrealistic.
- using the same B.Cs, we get
  $$\frac{B}{A} = \frac{k_1 - i\kappa_2}{k_1 + i\kappa_2} \qquad \frac{C}{A} = \frac{2k_1}{k_1 + i\kappa_2}$$
  ↳ note that $|B|^2 = |A|^2$, ∴ fully reflected
- but the wave does penetrate a bit.

Quantum tunnelling



- If an electron beam hits a finite boundary with $E < V_0$, there is a wave function beyond the boundary
  - ↳ i.e nonzero prob of electron tunnelling
  - ↳ prob depends on E and barrier width.

Waves in boxes

- The 3D time-independent Schrödinger equation:

$$\nabla^2 \psi + \frac{2m}{\hbar^2}(E - V)\psi = 0$$

- In a box we can separate the components, $\hat{\psi} = \hat{X}(x)\,\hat{Y}(y)\,\hat{Z}(z)$ then solve in each dimension.
- The solutions are a standing wave:

$$X(x) = A_x \sin k_x x \qquad Y(y) = A_y \sin k_y y \qquad Z(z) = A_z \sin k_z z$$

$$k_x = \frac{n_x \pi}{a} \qquad k_y = \frac{n_y \pi}{b} \qquad k_z = \frac{n_z \pi}{c}$$

- The energy levels are: $E_{n_x, n_y, n_z} = \frac{\pi^2 \hbar^2}{2m}\left(\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right)$
- This leads to degeneracy: states with different quantum numbers but the same energy.
- Normalising requires $\iiint |\psi|^2\, dx\,dy\,dz = 1$.

- In reality, spherical wells are more common, e.g the H atom with $V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}$.
  - ↳ in this case: $E_n = \frac{-13.6}{n^2}$ eV