Vectors

• Combining a vector with its additive inverse gives the zero vector, with length 0 and undefined direction.
• A scalar product projects one vector onto another.
• We can resolve \( \vec{a} \) into \( \parallel \) and \( \perp \) vectors w.r.t. some \( \hat{a} \)
  \[ \vec{a} = \vec{a}_{\parallel} - (\vec{a} \cdot \hat{a}) \hat{a} \]
• Distributive property of dot product can be proved diagrammatically.
• Derive cosine rule with \( |\vec{a}| - |\vec{a} + \vec{k}| \Rightarrow |\vec{a}|^2 = (\vec{a} + \vec{k}) \cdot (\vec{a} + \vec{k}) \).

Vector product

• \( \vec{a} \times \vec{k} = |\vec{a}| |\vec{k}| \sin \theta \hat{n} \left\langle \text{only unique in 3D.} \right\rangle 
• \( \vec{a} \times \vec{k} = -\vec{k} \times \vec{a} \) (anticommutative)
• \( \vec{a} \times \vec{k} = 0 \Rightarrow \vec{a} \parallel \vec{k} \) OR \( \vec{a} \) or \( \vec{k} = 0 \).
• \( |\vec{a} \times \vec{k}| \) is the area of a parallelogram.
• Non-associative; i.e., \( \vec{a} \times (\vec{k} \times \vec{c}) \neq (\vec{a} \times \vec{k}) \times \vec{c} \).

Vector area

• Vector area \( \mathcal{S} \) of a finite plane surface is defined such that \( |\mathcal{S}| = \text{area} \), with \( \mathcal{S} \) pointing normal to surface.
• The area of a projection (e.g., onto xy-plane) is \( \mathcal{S} \cdot \hat{z} \).
• We can define a total vector area for a composite surface as the sum of vector area elements, \( \mathcal{S} = \sum \mathcal{S} \)
  \( \Rightarrow \sum \mathcal{S} = 0 \) for a closed surface = 0.

Triple products

• Scalar triple product: \( \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a \cdot (b \times c) \)
  \( \left\langle \text{invariant under cyclic permutation, i.e., } a \cdot (b \times c) = c \cdot (a \times b) = b \cdot (c \times a) \right\rangle \)
  \( \left\langle \text{gives the volume of a parallelepiped} \right\rangle \)
If scalar triple product is zero, vectors are coplanar.

The vector triple product is $a \times (b \times c)$, which can be evaluated with the BAC-CAB rule:

$$a \times (b \times c) = b (a \cdot c) - c (a \cdot b),$$

$\Rightarrow a \times (b \times c)$ lies in the plane of $b$ and $c$.

Lines and planes

A line is parameterised by $\lambda$: $\mathbf{r} = a + \lambda \mathbf{t}$

Because $(\mathbf{r} - a) \parallel \mathbf{t}$, we can also write: $\mathbf{r} \times \mathbf{t} = a \times \mathbf{t}$

For a plane: $\mathbf{r} = a + \lambda \mathbf{f} + \mu \mathbf{g}$

$\Rightarrow \mathbf{r} \cdot \mathbf{n} = a \cdot \mathbf{n} = d$

$\Rightarrow$ the shortest distance to the origin is $|d|$.

Orthogonal basis

In 3D, any 3 non-coplanar vectors constitute a basis.
- Basis spans the space, i.e. $\mathbf{r} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c}$ where the components $\{ \lambda, \mu, \nu \}$ are unique.
- Basis vectors will have linear independence.

Components can be extracted using the reciprocal basis

Cyclic order preserved. $\begin{Bmatrix} \mathbf{A} = \frac{\mathbf{b} \times \mathbf{c}}{[a,b,c]} \\ \mathbf{B} = \frac{\mathbf{c} \times \mathbf{a}}{[a,b,c]} \\ \mathbf{C} = \frac{\mathbf{a} \times \mathbf{b}}{[a,b,c]} \end{Bmatrix}$

$\Rightarrow$ the component is just dot product of $\mathbf{r}$ with the appropriate reciprocal basis vector:

$$\mathbf{\lambda} = \mathbf{A} \cdot \mathbf{r} \quad \mathbf{\mu} = \mathbf{B} \cdot \mathbf{r} \quad \mathbf{\nu} = \mathbf{C} \cdot \mathbf{r}$$

A basis is orthonormal if all basis vectors are $\mathbf{b}$ and have unit length.

Right-handed if $[a,b,c] > 0$.

Direction cosines are cosines of angles between $\mathbf{a}$ and coordinate ones, i.e. $a = |a| (\cos \theta_x, \cos \theta_y, \cos \theta_z)$ in Cartesian.
In Cartesian, $\mathbf{a} \cdot \mathbf{b}$ is invariant under rotation.

$$\mathbf{a} \times \mathbf{b} = \left| \begin{array}{ccc}
1 & i & j \\
j & a_y & a_z \\
k & a_z & a_x
\end{array} \right|$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \left| \begin{array}{ccc}
1 & a_y & a_z \\
\mathbf{b} \times \mathbf{c} & a_x \\
\mathbf{c} \times \mathbf{a} & a_y
\end{array} \right| = \text{i.e. transformed volume of a unit cube.}$$

**Polar coordinates**

- Point specified by $(r, \phi)$
  $$x = r \cos \phi \quad y = r \sin \phi$$
  $$r = \sqrt{x^2 + y^2} \quad \phi = \tan^{-1} \left( \frac{y}{x} \right).$$

- Circle described by $r = a$

- Straight line at angle $\alpha$ to y-axis with shortest dist tol:
  $$r \cos (\phi - \alpha) = a.$$

- We can use the following orthonormal basis:
  $$\hat{e} = \cos \phi \hat{i} + \sin \phi \hat{j}$$
  $$\hat{\phi} = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

- We can evaluate $\hat{e}$:
  $$\hat{e} = \hat{r}^2 + r \hat{\phi} \hat{\phi}$$

- The area element will be $r \, r \, d\phi$.

**Cylindrical coordinates**

- Extension of plane polar coordinates to include $z$.
  $$x = r \cos \phi \quad y = r \sin \phi \quad z = z$$

- Volume element is:
  $$dV = r \, dr \, d\phi \, dz.$$
Spherical coordinates

Points described by radius, polar angle, azimuthal angle (i.e. \( r, \theta, \phi \)).

\[
\begin{align*}
x &= rsin\theta cos\phi \\
y &= rsin\theta sin\phi \\
z &= rcos\theta \\
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \text{cos}^{-1}\left(\frac{z}{r}\right) \\
\phi &= \text{tan}^{-1}\left(\frac{y}{x}\right)
\end{align*}
\]

We can find the orthogonal basis vectors using:

\[
\begin{align*}
\hat{r} &= \frac{\partial r}{\partial r} / \left| \frac{\partial r}{\partial r} \right| \\
\hat{\theta} &= \frac{\partial r}{\partial \theta} / \left| \frac{\partial r}{\partial \theta} \right| \\
\hat{\phi} &= \frac{\partial r}{\partial \phi} / \left| \frac{\partial r}{\partial \phi} \right|
\end{align*}
\]

\[
\begin{align*}
\hat{r} &= \sin\theta cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k} \\
\hat{\theta} &= \cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k} \\
\hat{\phi} &= -\sin\phi \hat{i} + \cos\phi \hat{j}
\end{align*}
\]

\[
\text{d}V = (dr)(r \text{d}\theta)(r \sin\theta \text{d}\phi) = r^2 \sin\theta \text{d}r \text{d}\theta \text{d}\phi.
\]
Complex numbers

- Complex numbers are a closed field \( \Rightarrow \) all operations return \( z \)
- Complex conjugate \( z^* = a - ib \) for \( z = a + ib \)
  \( \Rightarrow \) \( z \cdot z^* = a^2 + b^2 \geq 0 \)
  \( \Rightarrow \) \( z + z^* = 2 \text{Re}(z) \)
  \( \Rightarrow \) \( z - z^* = 2i \text{Im}(z) \)
  \( \Rightarrow \) \( \frac{1}{z} = \frac{z^*}{|z|^2} \)
- Multiplying corresponds to scaling and rotation.
- Euler's identity: \((a \cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\)
  \( \Rightarrow \) can be used to derive trig identities
  e.g. \( \cos n\theta = \text{Re}(e^{i\theta})^n = \text{Re}(e^{i\theta + i\theta}) \)
  e.g. \( \cos \theta = \frac{1}{2}(z + z^{-1}) \Rightarrow \cos 5\theta = \frac{1}{2^5}(z + z^{-1})^5 \).
- Euler's formula: \( e^{i\theta} = \cos \theta + i \sin \theta \).
- The \( n \)th roots of unity are the solutions to \( z^n = 1 \) for positive \( n \).
  \( e^{i\theta} = 1 \Rightarrow \theta = \frac{2\pi k}{n}, \quad k = 0, 1, 2, \ldots, n-1 \)
  \( \Rightarrow \) roots are \( 1, \omega, \omega^2, \ldots, \omega^{n-1} \) with \( \omega = e^{2\pi i/n} \).
- We define the complex logarithm as:
  \( \ln z = \ln(\text{re}^{i\theta}) = \ln r + i(\theta + 2\pi n) \quad n = 0, \pm 1, \pm 2, \ldots \)
  \( \Rightarrow \) the principal value is \( \ln r + i\theta \) for \( \theta \in [0, 2\pi) \).
- Likewise, general powers will be multi-valued
  \( z^{2^n} = e^{2^n \ln z} \)
- The fundamental theorem of algebra states:
  \( a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0 \)
  has \( n \) complex roots for all possible complex coefficients.
Hyperbolic Functions

- **Define:** \( \cos z = \frac{1}{2}(e^{iz} + e^{-iz}) \) and \( \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \).
- The hyperbolic functions are these functions evaluated on the imaginary axis:
  \[
  \cosh y = \cos(iy) = \frac{1}{2}(e^y + e^{-y}) \\
  \sinh y = \frac{1}{i} \sin(iy) = \frac{1}{2i}(e^y - e^{-y})
  \]

We can then define \( \tanh, \sech, \cosech \) etc.

- We can generate identities by substituting \( iy \) in and using
  \[
  \cos iy = \cosh y, \quad \sin iy = i\sinh y.
  \]
  \[
  \Rightarrow \cosh^2 y - \sinh^2 y = 1
  \]
  \[
  \Rightarrow \cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B
  \]
- Inverse hyperbolic functions can be expressed as elementary functions.
Calculus and Analysis

Limits

1. Intuitively, \( \lim_{x \to x_0} f(x) = k \) means \( f(x) \) can be made arbitrarily close to \( k \) by making \( x \) close enough to \( x_0 \).

2. The \( \varepsilon-\delta \) definition: For real \( f(x) \) defined on some open interval containing \( x_0 \) (but not necessarily at \( x_0 \)), \( \lim_{x \to x_0} f(x) = k \) means for any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that:
   \[ |f(x) - k| < \varepsilon \text{ for all } 0 < |x - x_0| < \delta \]

   "I.e. if you give me an \( \varepsilon \), I can find \( \delta \) to stay within \( \varepsilon \) of \( k \).

3. In practice, we guess the limit then prove with \( \varepsilon-\delta \).

4. Limits at infinity: \( \lim_{x \to \pm \infty} f(x) = k \) for all \( x \geq X \).

5. Limits can be manipulated by addition and multiplication.

If a quotient is indeterminate (top and bottom both 0 or \( \pm \infty \)), we can use L'Hôpital's rule:
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.
\]

Continuity and Differentiability

1. A real function \( f(x) \) is continuous at \( x = a \) iff:
   i) \( f(a) \) exists
   ii) \( \lim_{x \to a} f(x) \) exists and equals \( f(a) \).

2. A function \( f(x) \) is differentiable at \( x = a \) iff:
   i) it is continuous at \( x = a \)
   ii) \( f'(a) \) exists i.e. \( \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \) exists.
Leibniz formula

- Used to find $n$th derivative of a product of functions (just like Binomial theorem):
  \[
  \frac{\partial^n(fg)}{\partial x^n} = \sum_{m=0}^{n} \binom{n}{m} f^{(n-m)}(x) g^{(m)}(x)
  \]
  \[= f^{(n)} g + n f^{(n-1)} g + \frac{n(n-1)}{2} f^{(n-2)} g^2 + \ldots + f g^{(n)} \]
- Can be proved by induction.

Infinite Series

- Given a sequence of terms $u_0, u_1, u_2, \ldots$, the $n$th partial sum is $S_n = \sum_{k=0}^{n} u_k$

- If the partial sums have a finite limit as $n \to \infty$, the infinite series is convergent.
  \[\Rightarrow\] if it doesn't converge, it either diverges or oscillates.

- If $\sum_{k=0}^{\infty} |u_k|$ converges, the series is absolutely convergent (which also implies $\sum_{k=0}^{\infty} u_k$ converges)
  \[\Rightarrow\] otherwise if $\sum_{k=0}^{\infty} u_k$ converges but $|u_k|$ doesn't, series is conditionally convergent.
  \[\Rightarrow\] for absolutely convergent series we can rearrange terms.

Geometric progressions

- $S_n = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r}$ because $S_n = r + r^2 + \ldots + r^k = r^{n+1} + S_n - 1$.

- Series is absolutely convergent for $|r| < 1$
  \[\Rightarrow\] $\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$

- If $|r| \geq 1$, series cannot converge.
Convergence tests

1. $u_k \to 0$ as $k \to \infty$ is a necessary condition for convergence (but insufficient, e.g. harmonic series).

2. Comparison test:
   - Compare with a series of known convergence, $v_k$
   - If all terms $\leq v_k$ for all $k \geq K$, $S_n$ converges
   - If all terms $\geq v_k$ for divergent $v$, $S_n$ diverges.
   - Try to compare with geometric series or harmonic series.
   - $p$-series test: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ by comparison with geometric series.
     $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges for $p \leq 1$ by comp. with harmonic

3. Ratio test
   - If $\lim_{k \to \infty} \frac{u_{k+1}}{u_k} < 1$, $S_n$ converges
   - If $\lim_{k \to \infty} \frac{u_{k+1}}{u_k} > 1$, $S_n$ diverges
   - If ratio = 1, test indeterminate.

4. Alternating series:
   - Use the Leibniz criterion.
     $\sum_{k=0}^{\infty} (-1)^k a_k$ with $a_k > 0$ converges if $a_k$ is monotonic decreasing for large enough $k$ and $\lim_{k \to \infty} a_k = 0$.

5. Integral test:
   - If $f(n)$ is continuous, positive, and decreasing on $[1, \infty)$:
     $\sum_{n=1}^{\infty} f(n)$ converges/diverges as $\int_1^{\infty} f(x) dx$. 
Power series

- Series of the form \( f(x) = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots \)
- Either:
  - converges for \( x = 0 \) only
  - converges for all finite \( x \)
  - converges for \( |x| < R \), diverges for \( |x| > R \).
- Using ratio test and \( L = \lim_{k \to \infty} \frac{a_{k+1}}{a_k} \)
  - convergent for \( |x| < \frac{1}{L} \), divergent for \( |x| > \frac{1}{L} \).
- For a complex power series, this will define a circle of convergence.

Taylor series

- \( f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \ldots \)
- Or Maclaurin series when \( a = 0 \): \( f(x) = f(0) + xf'(0) + x^2 f''(0) + \ldots \)
- We can truncate the Taylor series and add a remainder term:
  \( f(x) = f(0) + x f'(0) + \frac{x^2 f''(0)}{2} + \cdots + \frac{x^n f^{(n)}(0)}{n!} + R_n \)
  \( \text{with } R_n = \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t) \, dt \)
- \( L \) derived by \( f(x) = f(0) + \int_0^x f'(t) \, dt \) (F-TC) then LBP.

- \( e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \)
- \( \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \)
- \( \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \)
- \( \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \)
Newton-Raphson

- Helps us find $x^*$ such that $f(x^*) = 0$

- If we have an initial guess $x_0$, we need $h$ such that $f(x_0 + h) = 0$. 
  
  $0 = f(x_0 + h) \approx f(x_0) + hf'(x_0)$  

  $\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

- Then we can iterate this to converge on $x^*$.

- If $E_i$ is the error in $x_i$:
  
  $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \Rightarrow E_{i+1} = E_i - \frac{E_i f(x_i)}{f'(x_i)}$.

  $\Rightarrow$ approximating the last term with a Taylor expansion:

  $E_{i+1} \approx E_i \cdot \frac{f''(x^*)}{2f'(x^*)}$

  - i.e. rapid quadratic convergence

- If there is a turning point between the root and $x_i$, it may not converge.
Integration

- Formally: \( \int_a^b f(x) \, dx = \lim_{N \to \infty} \sum_{i=0}^{N-1} f(x_i) (x_{i+1} - x_i) \). \( \approx \) area under curve.

- Hyperbolic substitutions:
  \[
  \sqrt{x^2 + a^2} \Leftarrow x = a \sinh y \\
  \sqrt{x^2 - a^2} \Leftarrow x = a \cosh y \\
  \frac{a^2}{x^2} \Leftarrow x = a \tanh y
  \]

- Integrate using complex numbers, e.g., \( \int \cos x e^{x \phi} \, dx = Re\left( \int e^{x(x+i\phi)} \, dx \right) \).

- If \( I(a) = \int_{a(x)}^{b(x)} f(x; a) \, dx \)

\[
I'(a) = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial a} \, dx + \frac{\partial b}{\partial a} f(b; a) - \frac{\partial a}{\partial a} f(a; a)
\]

Stirling's approximation

\[
\ln n! = \sum_{k=1}^{\infty} \ln n. \quad \text{But } \sum_{k=1}^{n} \ln x \, dx \leq \sum_{k=1}^{n} \ln n \leq \int_{1}^{n+1} \ln x \, dx
\]

\[
\therefore \ln n! \approx n \ln n - n \quad \text{for large } n.
\]

- Cauchy-Schwarz inequality

\[
\langle a, b \rangle^2 \leq \|a\|^2 \|b\|^2 \text{ where } \langle \cdot, \cdot \rangle \text{ is the inner product.}
\]

- For an \( N \)-dimensional vectors \( \langle a; b \rangle = \sum_{i=1}^{N} a_i b_i \)

\[
\left( \sum_{i=1}^{N} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{N} a_i^2 \right) \left( \sum_{i=1}^{N} b_i^2 \right)
\]

- Taking \( N \to \infty \), we get Schwarz's inequality

\[
\left( \int_{a}^{b} f(x)g(x) \, dx \right)^2 \leq \int_{a}^{b} (f(x))^2 \, dx \int_{a}^{b} (g(x))^2 \, dx.
\]
Multiple integrals

\[ \iiint f(x) \, dV = \lim_{N \to \infty} \sum f(x) \, dV. \]

- Cartesian: \( dV = dx \, dy \, dz \)
- Cylindrical: \( dV = r \, dr \, d\phi \, dz \)
- Spherical: \( dV = r^2 \sin \theta \, dr \, d\phi \, d\theta \)

We can do the integrals in any order.

If limits are independent, we can factor the integral out.

Gaussian distribution integrals

- \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \) is a common improper integral.
- Evaluate with polar coordinates:
  \[ I^2 = (\int_{-\infty}^{\infty} e^{-x^2} \, dx) (\int_{-\infty}^{\infty} e^{-y^2} \, dy) = \iint e^{-(x^2+y^2)} \, dxdy. \]
- Technically should use \( a \) in limits then \( \lim_{a \to \infty} \)
Probability

- Outcomes $w_i$ are mutually exclusive
- The sample space is the set of all possible outcomes: $\Omega = \{w_i\}$
- An event is a subset of $\Omega$

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \Rightarrow \quad P(A \mid B) = \frac{P(B \mid A) P(A)}{P(B)} \]

(Bayes' Theorem)

- Law of total probability: $P(A) = \sum_i P(A \mid B_i) P(B_i)$

Random variables

- Map sample states to an allowed value of the random variable, such that the subsets partition the space.
- Assign a probability distribution $P(x)$.

- Poisson distribution: $P(x = n) = e^{-\lambda} \frac{\lambda^n}{n!}$

\[ \lambda \] can be shown that it is the limit of a binomial distribution as $n \to \infty$, with $np = \lambda$.

- For continuous random variables, the probability density function is

\[ f(x) \, dx = P(x - dx \leq X < x + dx) \]

\[ P(a \leq X \leq b) = \int_a^b f(x) \, dx \quad \text{with} \quad \int_{-\infty}^{\infty} f(x) \, dx = 1. \]

\[ F(a) = \int_{-\infty}^{a} f(x) \, dx. \]

- Median is a $a$ such that $F(a) = \frac{1}{2}$
- Variance of a distribution is the same even when conditioned...
**Ordinary Differential Equations**

- A first-order ODE has the form \( F(y', y, x) = 0 \).
- An \( n \)-th order ODE: \( F(y^{(n)}, y^{(n-1)}, \ldots, y', y, x) = 0 \).
- A separable 1st order ODE:
  \[
  \frac{dy}{dx} = \frac{F(x)}{g(y)} \Rightarrow g(y)dy = f(g(x))dx.
  \]
- The general solution (including a constant) can be found by an initial/boundary condition.

- A linear 1st order ODE:
  \[
  \frac{dy}{dx} + p(x)y = f(x) \quad \text{if } f(x) = 0 \text{, it is homogeneous; and separable.}
  \]

  \( y \) and \( \frac{dy}{dx} \) appear linearly.

  \( \Rightarrow \) can be solved with an integrating factor, \( \mu(x) \), such that

  \[
  \mu(x) \cdot \text{LHS is the derivative of something } w.r.t. x.
  \]

  \[
  \Rightarrow \mu(x) = e^{\int p(x)dx}
  \]

  \( \Rightarrow \) \[
  \frac{\partial}{\partial x} (\mu(x)y) = \mu(x)f(x) \quad \text{which is easy to solve.}
  \]

- Substitutions may be required to make an ODE linear/separable.
- Homogeneous ODE:
  \[
  \frac{dy}{dx} = F\left(\frac{y}{x}\right) \quad \text{ie} \ F \text{ invariant when } x \text{ and } y \text{ scaled.}
  \]

  \( \Rightarrow \) solve by sub \( u = \frac{y}{x} \)

  \[
  \Rightarrow y = u(x)x \Rightarrow x \frac{du}{dx} + u = f(u) \quad \text{\(<\text{separable.}\)}
  \]

- Bernoulli ODE:
  \[
  \frac{dy}{dx} + p(x)y = q(x)y^n
  \]

  \( \Rightarrow \) sub \( z = y^{1-n} \)

  \[
  \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}
  \]

  \[
  \Rightarrow \frac{dz}{dx} = (1-n)[-p(x)z + q(x)] \quad \text{< linear.}
  \]
Second-order equations

- A linear 2nd order ODE: \( \frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = f(x) \).

Let with the linear differential operator \( L \), we can rewrite
\[
L = \frac{d}{dx}^2 + p(x) \frac{d}{dx} + q(x) \Rightarrow Ly = f(x)
\]

\( \Rightarrow L(xu) = xL(u) \) if \( x \) constant, \( \Rightarrow \) because linear.

\( \Rightarrow L(u+v) = L(u) + L(v) \)

- For a homogeneous 2nd order ODE \( (Ly = 0) \), any linear combination of solutions is a solution by principle of superposition.
- For inhomogeneous case, i.e. \( Ly = f(x) \):
  - A particular integral is any solution of \( Ly = f(x) \)
  - The complementary function \( y_c \) is the general solution of \( Ly = 0 \)
  - The general solution is the sum: \( y(x) = y_c(x) + y_p(x) \).

2nd order ODEs are generally hard to solve unless constant coefficients.

Consider homogeneous 2nd order linear ODE:
\[
\frac{d^2y}{dx^2} + 2a \frac{dy}{dx} + by = 0.
\]

Let sub \( y = e^{\lambda x} \) as a trial gives the auxiliary equation \( \lambda^2 + 2a \lambda + b = 0 \)

\( \Rightarrow \) if roots are negative real, we have oscillatory behavior.

\( \Rightarrow \) if \( \lambda_1 = \lambda_2 \), we have critical damping: \( y = (C_1 + C_2 x)e^{-ax} \).

- For linear 2nd order inhomogeneous ODEs with constant coefficients:
  - \( y_c \) can be found as above.
  - \( y_p \) can be found with trial solutions
    - If \( f(x) \) is a polynomial, try \( y_p = \text{polynomial of same degree} \).
    - If \( f(x) = xe^{\lambda x} \), try \( y_p = de^{\lambda x} \).
    - If \( f(x) = \cos kx + \sin kx \), try \( y_p = d_1 \cos kx + d_2 \sin kx \).
  - But if scalar multiples of these trial solutions are already solutions of the homogeneous eq, we may need to multiply by \( x \) or \( x^2 \) and try again.
Alternatively, since it is linear and differential operators commute, we can factorise:

\[(\frac{d}{dx} - \lambda_1)(\frac{d}{dx} - \lambda_2) = f(x)\]

Let \(z(x) = (\frac{d}{dx} - \lambda_2)y \Rightarrow (\frac{d}{dx} - \lambda_1)z = f(x)\).

Let solve for \(z\) then for \(y\).

This gives us a particular integral.
Multivariable calculus

- Mixed partial derivatives are always equal, and partial derivatives commute: \( f_{xy} = f_{yx} \)
- Integrating w.r.t one variable, we can treat others as constant but then we will need to add an arbitrary function.
- For \( f(x, y) \), \( df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \).
- Taylor series becomes:
  \[
  f(x+h, y+k) = f(x, y) + f_x(x, y)h + f_y(x, y)k + \frac{1}{2} f_{xx}h^2 + f_{xy}hk + \frac{1}{2} f_{yy}k^2 + \ldots
  \]

- Suppose \( f(x, y) \) where \( x = x(u, v), y = y(u, v) \). By an abuse of notation, we write \( f(x, y) = f(u, v) \) even though they are different functions:

  \[
  \begin{align*}
  \frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \quad \text{multivariable chain rule=} \\
  \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}
  \end{align*}
  \]

  e.g. \( f(x, y) \to f(r, \theta) \) : \( x = r \cos \theta, y = r \sin \theta \)
  \[
  \frac{\partial f}{\partial r} = \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \quad \text{etc.}
  \]

- If both \( x \) and \( y \) are functions of \( t \):
  \[
  \frac{df}{dt} = \left( \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right)
  \]

- If we have \( f(x, y, z) = 0 \), then the partial derivatives have reciprocity and are cyclic.
  i.e. \( \left( \frac{\partial x}{\partial y} \right)_z = \left( \frac{\partial y}{\partial x} \right)_z \) and \( \left( \frac{\partial x}{\partial z} \right)_y \left( \frac{\partial y}{\partial x} \right)_z \left( \frac{\partial x}{\partial x} \right)_z = -1 \)
Exact differentials

\[ w = p(x,y) \, dx + q(x,y) \, dy \] is a differential form in \( x \) and \( y \).

\( w \) is an exact differential if \( \exists F(x,y) \) such that \( df = p \, dx + q \, dy \).

\( \Rightarrow \) equivalently, exact iff \( \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x} \).

\( \Rightarrow \) if \( p \, dx + q \, dy \) is exact, \( F(x,y) = c \).

\( \Rightarrow \) we can make an exact differential form exact with an integrating factor: \( \mu(x,y) \cdot [p \, dx + q \, dy] \).

\( \Rightarrow \) this is very difficult to solve for \( \mu \), so we instead try to find \( \mu(x) \) or \( \mu(y) \) only.

\( \Rightarrow \) e.g. \( \mu(x) \):

\[ \mu \frac{\partial p}{\partial y} = q \frac{\partial \mu}{\partial x} + \mu \frac{\partial q}{\partial x} \] if exact

\[ \Rightarrow \frac{1}{\mu} \frac{d \mu}{dx} = \frac{1}{\alpha} \left( \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} \right). \]

\[ \Rightarrow \text{likewise for } \mu(y): \frac{1}{\mu} \frac{d \mu}{dy} = -\frac{1}{\rho} \left( \frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} \right). \]

Maxwell's relation

\( \Rightarrow \) Any two of \( (p, V, T, S) \) can describe the state of a gas.

\( \Rightarrow \) Given a thermodynamic relation, we

\( \Rightarrow \) The fundamental thermodynamic relation is

\[ \frac{dV}{T} = -p \, dV \]

\( \Rightarrow \) if we treat \( V \) as a function of \( (S, V) \):

\[ \frac{dV}{T} = \left( \frac{\partial V}{\partial S} \right)_V \, dS + \left( \frac{\partial V}{\partial V} \right)_S \, dV \]

\[ \Rightarrow \left( \frac{\partial V}{\partial S} \right)_V = T \text{ and } \left( \frac{\partial V}{\partial V} \right)_S = -p \]

\[ \Rightarrow \left( \frac{\partial V}{\partial S} \right)_S = -\left( \frac{\partial p}{\partial S} \right)_V \text{ by mixed partials. This is one of } \text{Maxwell's relations.} \]

\( \Rightarrow \) We can derive the others using Legendre transformations

\[ \Rightarrow F = V - TS \Rightarrow dF = -SdT - p \, dV \]

\[ \Rightarrow H = V + PV \Rightarrow dH = TdS + Vdp \]

\[ \Rightarrow G = H - TS \Rightarrow dG = -SdT + Vdp. \]
We can also derive a different type of relation:
\[ dV = T dS - P dV \]
but let \( U = U(T, S) \)

\[ \Rightarrow \Delta U = T dS - P \left[ \frac{\partial V}{\partial T} dT + \frac{\partial V}{\partial S} dS \right] \]

then we take partial derivatives and equate.

Stationary points

- Because \( f(x) = f(x_0) + \nabla f(x_0) \cdot (x - x_0) \), if a point is stationary if \( \nabla f(x_0) = 0 \).
- To find the character of the stationary points, we use the determinant of the Hessian:
  \[ H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \]
  \[ \det H > 0 \quad \text{and} \quad f_{xx} > 0 \quad \Rightarrow \text{minimum} \]
  \[ \det H > 0 \quad \text{and} \quad f_{xx} < 0 \quad \Rightarrow \text{maximum} \]
  \[ \det H < 0 \quad \Rightarrow \text{saddle} \]
  \[ \det H = 0 \quad \text{inconclusive} \]
- For more variables:
  \[ \Rightarrow \text{if all eigenvalues} > 0, \text{ min} \]
  \[ \Rightarrow \text{if all eigenvalues} < 0, \text{ max} \]
  \[ \Rightarrow \text{else saddle} \]

Conditional stationary values

- To optimise \( f(x, y) \) subject to \( g(x, y) = c \), solve
  \[ \nabla f = \lambda \nabla g \]
  where \( \lambda \) is a Lagrange Multiplier
- Consider some displacement \( d\vec{x} \)
- \( d\vec{x} \) must be tangent to \( g(x, y) = 0 \).
  \[ \Rightarrow (\nabla g) \cdot d\vec{x} = 0 \]
- Likewise, \( d\vec{f} = (\nabla f) \cdot d\vec{x} = 0 \) by definition of a stationary point
- \[ \Rightarrow \nabla f \parallel \nabla g \]
- For more constraints:
  \[ \nabla f = \lambda \nabla g + \mu \nabla h \]
Boltzmann distribution

Consider a system which has \( n \) possible discrete states, in which holds \( N_i \) particles whose energy is \( E_i \):

\[ N = \sum_{i=1}^{n} N_i \]

\[ E = \sum_{i=1}^{n} N_i E_i \]

A given distribution of particles can be achieved in \( W \) ways:

\[ W = \frac{N!}{N_1!N_2!...N_n!} \]

The most likely state maximises \( W \), or \( \ln W \) equivalently:

\[ \ln W = \ln(N!) - \sum_{i=1}^{n} \ln(N_i!) \]

\[ L = \ln(N!) - \sum_{i=1}^{n} \ln(N_i!) - \alpha \left( \sum_{i=1}^{n} N_i - N \right) - \beta \left( \sum_{i=1}^{n} N_i E_i - E \right) \]

\( N_i \) are the variables, \( \alpha, \beta \) need solving for.

\[ \frac{\partial L}{\partial N_i} = \ln N - \ln N_i - \alpha - \beta E_i \]

Then set \( \frac{\partial L}{\partial N_i} = 0 \) and solve for \( N_i \):

\[ N_i = Ne^{-\alpha - \beta E_i} \]

This gives the Boltzmann dist.

Different assumptions about particle states leads to different \( W \).
Vector calculus

- Let $\phi(x, y, z)$ be a scalar field.
  $$\text{grad } \phi = \nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$
- The rate of change of $\phi$ in direction $\mathbf{t}$ is the directional derivative
  $$\frac{d\phi}{ds} = \mathbf{t} \cdot \nabla \phi.$$ 
- This implies that $\mathbf{t} \cdot \nabla \phi$ is the direction of most rapid increase.
- Given a surface $F(x, y, z) = c$, $\nabla F$ must be normal to the surface because $F$ is constant along the surface.
  $$\Rightarrow \mathbf{n} = \frac{\nabla F}{|\nabla F|}.$$

Line integrals

- Consider a curve parameterized by $t$: $\mathbf{r}(t) = (x(t), y(t), z(t))$
  $$d\mathbf{r} = \frac{d\mathbf{r}}{dt} dt$$
- For a scalar field parameterized by an arc length $s$:
  $$\int_C \phi \, ds = \int_s \phi(\mathbf{r}(s)) \, ds$$
- For a more general parameter $t$:
  $$\int_C \phi \, ds = \int_t^s \phi(\mathbf{r}(t)) \left| \frac{d\mathbf{r}}{dt} \right| \, dt$$
- For a vector field $\mathbf{E}(t)$
  $$\int_C \mathbf{E} \cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{E}(\mathbf{r}(t)) \frac{d\mathbf{r}}{dt} \, dt.$$ 
- The Gradient theorem:
  $$\int_C (\nabla \phi) \cdot d\mathbf{r} = \int_{t_1}^{t_2} d\phi = \phi(t_2) - \phi(t_1)$$
Conservative fields

- A line integral independent of the path.
- \( \mathbf{E} = -\nabla \phi \) for some \( \phi(x) \)
- \( \mathbf{E} \cdot d\mathbf{s} \) is exact}
- \( \oint \mathbf{E} \cdot d\mathbf{s} = 0 \) for all closed curves.
- \( \nabla \times \mathbf{E} = 0 \)

Surface integrals

- For a general curved surfaces \( S \) in space, the vector area element is defined by \( d\mathbf{A} = \mathbf{n} \, dS \). The total vector area is \( \int_S \mathbf{n} \, dS \)
- The flux of \( \mathbf{E} \) through \( S \) is defined by:
  \[ \int_S \mathbf{E} \cdot d\mathbf{A} = \int_S \mathbf{E} \cdot \mathbf{n} \, dS \]

Divergence

- \( \text{div} \, \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \) \( \in \text{SCALAR} \)
- The divergence theorem:
  \[ \iiint_V (\nabla \cdot \mathbf{F}) \, dV = \iint_S \mathbf{F} \cdot d\mathbf{S} \]
- Can be used to define divergence:
  \[ \nabla \cdot \mathbf{E} = \lim_{\delta V \to 0} \frac{1}{\delta V} \int_{S_S} \mathbf{E} \cdot d\mathbf{S} \]
- If a surface is not closed, we can first construct a closed one then apply the divergence theorem.
- The Laplacian is the divergence of a gradient
  \[ \nabla^2 \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \]
  \( \Rightarrow \) it is also a scalar
\[ \nabla \times \mathbf{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ F_x & F_y & F_z \end{vmatrix} \quad \text{(Vector)} \]

- **Stokes theorem:** \[ \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = \oint_C \mathbf{E} \cdot d\mathbf{r} \]
  where \( C \) bounds \( S \).

  L.H.S. we RH grip rule for direction.

- This leads to a geometric definition: \[ \hat{n} \cdot (\nabla \times \mathbf{E}) = \lim_{h \to 0} \frac{1}{h} \int_{c} \mathbf{E} \cdot d\mathbf{s} \]
- For any vector conservative field \( \mathbf{E} = -\nabla \phi \), \( \nabla \times \mathbf{E} = 0 \).
- Many different surfaces can be bounded by a closed curve but only one volume is bounded by a closed surface.
- A multiply connected surface may have multiple bounding curves.

  e.g. annulus

\[ \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = \int_C \mathbf{E} \cdot d\mathbf{r} + \int_S \mathbf{E} \cdot d\mathbf{A} \]

- For a planar surface, we can use Green's theorem, a special case.
Fourier Series

- Functions are orthogonal on an interval if their inner product is zero:
  \[ \int_a^b f(x)g(x) \, dx = 0 \]

- On the interval \([-\pi, \pi]\), all \(\cos mx\) and \(\sin mx\), \(m,n \in \mathbb{Z}\) are mutually orthogonal (but not normalised):
  \[
  \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 
  2\pi, & m = n = 0 \\
  \pi, & m = n \neq 0 \\
  0, & m \neq n 
  \end{cases}
  \]
  \[
  \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 
  \pi, & m = n \neq 0 \\
  0, & \text{otherwise.}
  \end{cases}
  \]
  \[
  \int_{-\pi}^{\pi} \cos mx \sin nx \, dx = 0
  \]

- We can change bounds to \(\pm L\) provided we scale \(mx \to \frac{mt\pi}{L}\) then this will work for any \([a,b]\) such that \(2L = b - a\).
- \(\sin mx\) and \(\cos nx\) form a basis, such that almost any \(f(x)\) can be represented with a Fourier series:
  \[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{nt\pi x}{L} + b_n \sin \frac{nt\pi x}{L} \right) \]
  \[ L \]

- Fourier coefficients can be found by integrating and after multiplying with \(\cos \left( \frac{nt\pi x}{L} \right)\) or \(\sin \left( \frac{nt\pi x}{L} \right)\):
  \[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{nt\pi x}{L} \, dx \]
  \[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{nt\pi x}{L} \, dx \]

- For even functions, all \(b_n=0\), so it is a cosine series.
- For odd functions, all \(a_n=0\), so it is a sine series.
- Fourier coefficients decrease like \(1/n\), so we can approximate functions.
- We can observe how fast the coefficients decline to understand convergence.
- Around a discontinuity, the Fourier series will always overshoot, even in the limit, though the width of the overshoot decreases. Gilly phenomenon.
- Differentiating always reduces smoothness; Fourier coefficients drop less rapidly.
The mean-square value of a periodic function can be evaluated using Parseval's theorem:
\[
\frac{1}{2L} \int_{-L}^{L} |f(x)|^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)
\]

The set of values, for different \( n \), is the power spectrum and describes how power is distributed amongst the harmonics.

**Complex Fourier series**
\[
f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i \frac{2\pi nx}{L}}
\]
\[
c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i \frac{2\pi nx}{L}} dx
\]

*\( e^{i \frac{2\pi nx}{L} } \) is used as a basis.

* For complex functions, \( f(x) \) and \( g(x) \) are orthogonal if:
\[
\int_{a}^{b} [f(x)]^* g(x) dx = 0.
\]
**Linear Algebra**

- A **linear vector space** over a field of scalars defines addition and scalar multiplication: associative, commutative, distributive.
- A **mapping** of a vector space assigns \( x \in V \) to \( y \in V \)
  
  e.g. \( A: x \to y \) or \( A\mathbf{x} = \mathbf{y} \)

**Matrices**

- **Subscript notation:** \( A = (a_{ij}) \), \( (A)_{ij} = a_{ij} \)
- Unsummed indices must match.
- **Matrix addition** \( C = A + B \Rightarrow C_{ij} = a_{ij} + b_{ij} \)
- **Matrix mult (not commutative):** \( C_{ij} = A_{ik}b_{kj} \) (sum implied).
- The **commutator** is defined by \( C = [A, B] = AB - BA \)

- The **transpose** is given by \( (M^T)_{ij} = (M)_{ji} \)
  
  - \( (MT)^T = M \)
  - \( (ABC...YZ)^T = Z^T Y^T ... C^T B^T A^T \)
- A symmetric matrix satisfies \( S^T = S \), i.e. \( a_{ij} = a_{ji} \) (square matrices).
- An antisymmetric matrix satisfies \( A^T = -A \), i.e. \( a_{ij} = -a_{ji} \) (square matrices).
- We can always decompose a square matrix \( B \) into \( A \) and \( S \):
  
  \[ S = \frac{1}{2} (B + B^T) \quad A = \frac{1}{2} (B - B^T) \]

- A **diagonal matrix** has nonzero entries solely on the diagonal.
- The **identity matrix** has ones on the diagonal \( I = (d_{ij}) \)
- An **orthogonal matrix** is a square matrix that satisfies \( OO^T = O^TO = I \)
- The complex conjugate of a matrix: \( A^* = (a_{ij}^*) \)
- The **hermitian conjugate** is \( A^\dagger = (A^T)^* = (A^*)^T = (a_{ji}^*) \)
- The **trace** is the sum of diagonal elements: \( \text{tr} \ A = a_{ii} \)
  
  \( \Rightarrow \) invariant under cyclic permutation i.e. \( \text{tr} ABC = \text{tr} CBA \)
Determinants

- The minor of a matrix element is the matrix made by deleting the ith and jth rows. The cofactor is the signed minor.
- The classical adjoint of a matrix contains the transposed cofactors.
- The general rule for a determinant: \( |B| = \sum_{j=1}^{n} b_{ij} (\text{adj } B)_{ij} \)

\[ \Rightarrow \text{ expand on a (signed) row/col and compute sub-determinants.} \]

- Can be written in terms of the Levi-Civita tensor:
  \[ \varepsilon_{ijk} = \begin{cases} 0 & \text{any pair of } i, j, k \text{ equal} \\ 1 & \text{even permutation} \\ -1 & \text{odd perm.} \end{cases} \]

\[ \Rightarrow |A| = \varepsilon_{ijk} a_{1i} a_{2j} a_{3k} = \varepsilon_{ijk} a_{i1} a_{j2} a_{k3} \]

- From this we can derive some key properties:
  - Interchanging any two rows/cols flips sign of det
  - \( \det A = 0 \) if any two rows/cols are the same
  - \( \det(AB) = (\det A)(\det B) \)
  - \( \det A = \det A^T \)

Inverse

- If \( A^{-1} \) exists, it is both the left and right inverse:
  \[ A^{-1}A = AA^{-1} = I \]
- We can find the inverse using:
  \[ A^{-1} = \frac{\text{adj } A}{\det A} \]

- If \( \det A = 0 \), matrix is singular (i.e., no inverse)
- An orthogonal matrix satisfies \( OO^T = O^T O = I \)
  \[ \therefore O^{-1} = O^T \quad \text{and} \quad 10^T 10 = 10^2 = 1 \]
- Rotations and reflections are both orthogonal.
  - E.g., a rotation gives \( x' = x - 2(x \cdot n)n \)
  \[ \Rightarrow 0 = I - 2nn^T \]
Linear equations

- If $Ax = y$ and $|A| \neq 0$, we can use Cramer's rule:
  $$x_i = \frac{\det A_i}{\det A}$$
  where $A_i$ is $A$ with the $i$th column replaced by vector $y$.

- If $A$ and $y$ are shorter than $x$, system is underdetermined and we have a family of solutions that live in a subspace.

- If $A$ and $y$ are taller than $x$, we may have redundancy or inconsistency.

- If $A$ and $y$ are the same height as $x$:
  - $|A| \neq 0$ $\Rightarrow$ unique solution
  - $|A| = 0, y \neq 0$ $\Rightarrow$ not unique

Eigenvalues and eigenvectors

$n \times n$:

$A \mathbf{v} = \lambda \mathbf{v} \Rightarrow (A - \lambda I)\mathbf{v} = 0 \Rightarrow \det(A - \lambda I) = 0$.

- Eigenvalue
- Eigenvector

- The determinant is called the characteristic polynomial $p_\lambda(A)$, degree $n$.
- The set of eigenvalues is the spectrum of $A$.
- $\Rightarrow \lambda$ may be complex, corresponding to a rotation.
- Trace $= \sum$ of eigenvalues.
- Determinant $= \text{product}$ of eigenvalues.

- Eigenvectors can be found by solving $(A - \lambda I)\mathbf{v} = 0$.

- Real symmetric matrices (i.e., $A = A^T$) have real eigenvalues.

- The eigenvectors of a symmetric matrix are orthogonal.

- For a real symmetric matrix $A$, with orthonormal eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ as the columns, i.e., $X = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$, $X^TX = I$.

- $A^T X \mathbf{x} = (\lambda_1 \mathbf{e}_1 \ldots, 0)$.
Partial Differential Equations

- A general PDE has the form $F(x,y,..., f_x, f_y, ... f_{xx}, f_{xy}, f_{yy}, ... ) = 0$.
  - The order is the order of the highest derivative.
  - The wave equation: $\frac{\partial^2 \psi}{\partial t^2} = c^2 \nabla^2 \psi$

- In general, boundary conditions will be functions.

- In the heat equation, the rate of heating is proportional to the convexity of the temperature surface: $\frac{\partial \Theta}{\partial t} = \kappa \nabla^2 \Theta$

- In electrodynamics, $\nabla^2 \varphi = -\frac{\rho}{\varepsilon_0}$ (Poisson's equation), reduces to Laplace's equation if $\rho = 0$.

- The choices of B.Cs are:
  - Dirichlet condition: give the value of $\varphi$ on $\partial D$, e.g. to model heat propagation from boundary to interior.
  - Neumann condition: give the normal derivative of $\varphi$ on $\partial D$ e.g. to find potential after specifying the field.
  - Linear combination of the above.

- A general linear 2nd order PDE in 2D:
  $\alpha \frac{\partial^2 \psi}{\partial x^2} + 2\beta \frac{\partial^2 \psi}{\partial xy} + \gamma \frac{\partial^2 \psi}{\partial y^2} + ... + h \psi = 0$
  - Elliptic if $b^2 < 4ac$ e.g. Laplace's equation.
  - Parabolic if $b^2 = 4ac$ e.g. heat equation in 1D.
  - Hyperbolic if $b^2 > 4ac$ e.g. wave equation.
2D Elliptic and Hyperbolic PDEs

For equation of the form \( \frac{\partial^2 
abla^2 \Psi}{\partial x^2} + 2b \frac{\partial^2 
abla^2 \Psi}{\partial x \partial y} + c \frac{\partial^2 
abla^2 \Psi}{\partial y^2} = 0 \)

We try solutions \( \Psi(x, y) = f(x + py) = f(x) \).

From the chain rule, \( \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \), \( \frac{\partial f}{\partial y} = pf \frac{\partial \frac{\partial f}{\partial z}}{\partial z} \)

\( \Rightarrow cp^2 + 2bp + a = 0 \), \( \Rightarrow p \) and \( p \) complex for elliptic

The general sol will be a linear comb. of independent solutions:

\( \Psi(x, y) = f(x + py) + g(x - py) \)

\( \Rightarrow F \) and \( g \) are arbitrary function decided by the B.C.

* e.g. \( \Psi(x, t) = f(x - ct) + g(x + ct) \) for the wave equation

* e.g. \( \Psi(x, y) = f(x + iy) + g(x - iy) \)

\( \Rightarrow \) only need to use real part i.e. \( \Psi(x, y) = \text{Re} \{ f(z) + g(z^*) \} \)

Separation of variables

If we substitute \( \Psi(x, y) = X(x) Y(y) \), we end up with ODEs

Requires \( b = 0 \), if not change variables to \( \omega = x + ay \), \( z = x - by \).

After \( \Psi(x, y) = X Y \) and rearrange, we will have

\( F(X) = G(Y) \), thus they must equal a constant, \( \lambda \).

\( \Rightarrow \) For each allowed \( \lambda \), we will have a different \( X \) and \( Y \)

\( \Rightarrow \) general solution will be linear combination \( \Psi = \sum_{\lambda} \alpha \Psi_{\lambda}(x, y) \)