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The Calculus of Variations

- A functional maps a function to a value, e.g

$$G[y] = \int_\alpha^\beta f(y, y'; x)\,dx$$

- The calculus of variations can be used to extremise functionals.
- The variation of G is defined to be:

$$\delta G = G[y+\delta y] - G[y] = \int_\alpha^\beta \delta y \frac{\delta G}{\delta y}\,dx$$

definition of functional derivative.

$$= \int_\alpha^\beta \left[\frac{\partial f}{\partial y}\delta y + \underbrace{\frac{\partial f}{\partial y'}(\delta y)'}_{\text{integrate by parts}}\right]dx \quad \text{to first order}$$

$$= \underbrace{\left[\frac{\partial f}{\partial y'}\delta y\right]_\alpha^\beta}_{} + \int_\alpha^\beta \delta y\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]dx$$

Fixed endpoints $\therefore \delta y(\beta) = \delta y(\alpha) = 0$

- The functional is stationary when $\delta G = 0$, resulting in the Euler-Lagrange equation

$$\Rightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

↳ in the special case where f has no x-dependence,

$$\frac{df}{dx} = \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{d^2y}{dx^2} + \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + y''\frac{\partial f}{\partial y'}$$

↳ sub in EL to give $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{d}{dx}\left(y'\frac{\partial f}{\partial y'}\right)$

↳ so when $\frac{\partial f}{\partial x} = 0$:

$$\boxed{f - y'\frac{\partial f}{\partial y'} = \text{const}} \quad \text{Beltrami identity}$$

Fermat's principle

- Fermat's principle states that light chooses a path of stationary time, or equivalently, stationary optical path length:

$$P = \int_A^B \mu(\underline{r})\,dl$$

refractive index

- For general 3D motion $P[y, z] = \int_{x_A}^{x_B} \mu(y, z)\sqrt{1 + y'^2 + z'^2}\,dx$

↳ apply EL for both $y(x)$ and $z(x)$.

- For sound waves, we need an expression for the acoustic path length.

Hamilton's principle

- Lagrangian mechanics examines the motion of a point in configuration space, described by generalised coordinates $\{q_i\}$
- The action S of a path is a functional of the Lagrangian $L = T - V$:

$$S = \int_{t_0}^{t_1} L(\{q_i\}, \{\dot{q}_i\}, \ldots; t)\,dt$$

- Hamilton's principle states that the path in configuration extremises S ('least action')

$$\Rightarrow \frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \qquad i = 1, \ldots, N$$

↳ if L does not explicitly depend on time,

$$L - \sum_{i=1}^N \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \text{const}$$

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Constrained variation

- To maximise $f(x,y,z)$ s.t. $g(x,y,z)=0$, we extremise without constraint $f-\lambda g$, where λ is a Lagrange multiplier.
- To extremise a functional $G[y]$ s.t. $P[y]=0$, we just extremise $G[y]-\lambda P[y]$ using the variational calculus.

- Eigenfunctions of the SL equation can be regarded as the extremals of a certain functional
  ↳ let $F[y] \equiv \langle y|\mathcal{L}y\rangle = \int_\alpha^\beta p(x)(y')^2 - q(x)y^2\,dx$
  $G[y] = \langle y|y\rangle_w = \int_\alpha^\beta w(x)y^2\,dx.$
  ↳ we can show that the functional derivatives are
  $\delta F/\delta y = 2\mathcal{L}y \qquad \delta G/\delta y = 2wy$
  ↳ consider the ratio $\Lambda[y] = F[y]/G[y]$

(quotient rule) $$\frac{\delta\Lambda}{\delta y} = \frac{1}{G}\left[\frac{\delta F}{\delta y} - \frac{F}{G}\frac{\delta G}{\delta y}\right] = \frac{1}{G}\left[2\mathcal{L}y - 2\Lambda wy\right]$$

☆ ↳ hence Λ is extremised with values λ where λ satisfies the SL eigenvalue problem $\mathcal{L}y = \lambda wy$

- This gives rise to the Rayleigh-Ritz method for estimating eigenvalues
  ↳ if $p(x)>0$, $q(x)\le 0$ such that $F[y]\ge 0$, then $\Lambda \ge 0$
  ↳ one of the extrema, λ_0, is then the absolute minimum
  $\Rightarrow \Lambda[y] \ge \lambda_0$, with equality for eigenfunctions.
  ↳ hence we may find an upper bound by substituting a trial function, since $\lambda_0 \le \Lambda[y_{trial}]$
- We may decide to use a LC of basis functions as the trial, or to have a function with another parameter in it.
  ↳ the trial basis functions should satisfy the B.Cs.
  ↳ we can improve the bound by differentiating with respect to the parameter, e.g. $\partial y_{trial}/\partial a$ for $y=e^{-ax^2}$.