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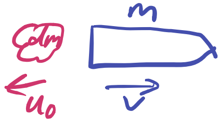
Classical Dynamics

Newtonian Mechanics

- We can write NII as: $m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} = \mathbf{F}$
  - ↳ e.g for a rocket in space, with no external forces:



$$u_0\,dm + m\,dv = 0 \Rightarrow v = u_0 \ln\left(\frac{m_i}{m_f}\right)$$

- The equation of motion for an object can be found by directly considering forces, or by differentiating E_{total}
- For a many-particle system, the centre of mass is

$\mathbf{R} = \frac{1}{M}\sum_i m_i \mathbf{r}_i$ ← capital letters for aggregate quantities.

↳ the total momentum $\mathbf{P}$ is changed by the total external force $\mathbf{F}_0$

$$\sum_a m_a \ddot{\mathbf{r}}_a = \sum_a \mathbf{F}_a = \overset{\text{external}}{\sum_a \mathbf{F}_{a0}} + \overset{\text{internal}}{\sum_a \sum_b \mathbf{F}_{ab}}$$ ← =0 due to NIII

$$\Rightarrow M\ddot{\mathbf{R}} = \mathbf{F}_0 \Rightarrow \dot{\mathbf{P}} = \mathbf{F}_0$$

↳ the total angular momentum $\mathbf{J}$ is changed by the total external torque $\mathbf{G}$

$$\sum_a \mathbf{r}_a \times \mathbf{p}_a = \sum_a \mathbf{r}_a \times \mathbf{F}_a = \sum_a \mathbf{r}_a \times \mathbf{F}_{a0} + \overbrace{\frac{1}{2}\sum_a \sum_b (\mathbf{r}_a - \mathbf{r}_b)\times \mathbf{F}_{ab}}^{\text{zero}}$$

$$\Rightarrow \dot{\mathbf{J}} = \mathbf{G}_0$$

- The kinetic energy of a particle is $T = \frac{1}{2}mv^2$

$$\mathbf{F}\cdot d\mathbf{r} = m\ddot{\mathbf{r}}\cdot d\mathbf{r} = m(\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}})dt = d\left(\frac{1}{2}mv^2\right)$$

↳ for a system of particles, this work may instead change the interaction between particles and increase the potential energy

↳ hence $E = T + U$ and $dE = \sum_a \mathbf{F}_{a0}\cdot d\mathbf{r}_a$

Coordinate systems

- Angular quantities depend on the choice of origin (obviously)

$$\mathbf{r} = \mathbf{r}' + \underset{\text{constant}}{\mathbf{a}} \Rightarrow \mathbf{J} = \mathbf{J}' + \mathbf{a}\times\mathbf{P}, \quad \mathbf{G} = \mathbf{G}' + \mathbf{a}\times\mathbf{F}$$

↳ the intrinsic angular momentum $\mathbf{J}'$ is defined in the zero momentum frame – it is independent of origin.

- Consider a Galilean transformation from $S' \to S$

$$\mathbf{r} = \mathbf{r}' + \mathbf{V}t, \quad t = t' \quad \text{(i.e nonrelativistic)}$$

↳ momentum is simple: $\mathbf{P} = \mathbf{P}' + M\mathbf{V}$

↳ angular momentum is $\mathbf{J} = \sum_a (\mathbf{r}_a' + \mathbf{V}t)\times(\mathbf{p}_a' + m_a\mathbf{V})$. If S' is the ZMF, the angular momentum is

$$\mathbf{J} = \mathbf{J}' + M\mathbf{R}'\times\mathbf{V}$$

↳ energy depends on the frame: $T = \underset{\text{KE in ZMF}}{T'} + \frac{1}{2}MV^2$

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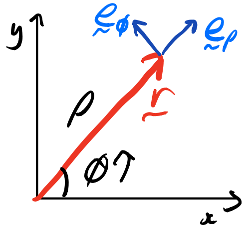
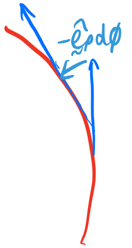
- Unit vector directions are only constant in Cartesian.
- Consider the dynamics in plane polars



↳ for general motion, ρ and ϕ are changing with ϕ, hence so are $\hat{\underline{e}}_\rho$ and $\hat{\underline{e}}_\phi$.

↳ but the unit vectors can only change orthogonal to themselves

$$\Rightarrow \boxed{\begin{aligned}\dot{\underline{e}}_\rho &= \dot{\phi}\,\hat{\underline{e}}_\phi \\ \dot{\underline{e}}_\phi &= -\dot{\phi}\,\hat{\underline{e}}_\rho\end{aligned}}$$

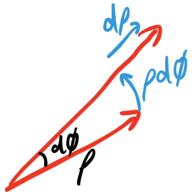


↳ the velocity can be derived directly by geometry

$$\underline{r} = \rho\hat{\underline{e}}_\rho + \phi\,\hat{\underline{e}}_\phi$$

$$\Rightarrow d\underline{r} = \rho\, d\phi\, \hat{\underline{e}}_\phi + d\rho\, \hat{\underline{e}}_\rho$$

$$\Rightarrow \boxed{\dot{\underline{r}} = \underbrace{\dot{\rho}\,\hat{\underline{e}}_\rho}_{\text{radial}} + \underbrace{\rho\dot{\phi}\,\hat{\underline{e}}_\phi}_{\text{tangential}}}$$



- Acceleration in plane polars is given by:

$$\boxed{\ddot{\underline{r}} = (\ddot{\rho} - \rho\dot{\phi}^2)\,\hat{\underline{e}}_\rho + (2\dot{\rho}\dot{\phi} + \rho\ddot{\phi})\,\hat{\underline{e}}_\phi}$$

↳ the radial term includes the centripetal acceleration

↳ the transverse term $= \frac{1}{\rho}\frac{d}{dt}(\rho^2\dot{\phi})$ — angular momentum per unit mass

- We can instead express polar coordinates on an Argand diagram, hence $\hat{\underline{e}}_\rho \rightarrow e^{i\phi}$, $\hat{\underline{e}}_\phi \rightarrow ie^{i\phi}$

$$\therefore \frac{d^2}{dt^2}(\rho e^{i\phi}) = (\ddot{\rho} - \rho\dot{\phi}^2)e^{i\phi} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})ie^{i\phi}$$

Rotating frames

- If there is a frame S_0 in which $m\ddot{\underline{r}}_0 = \underline{F}$, where $\underline{F}$ is generated by known physical causes, what is the equation of motion in a moving frame S?

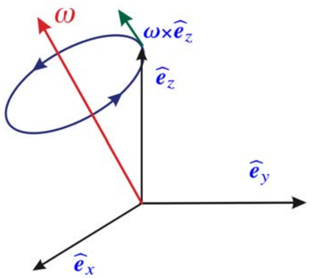
$$\underline{r} = \underline{r}_0 - \underline{R}(t) \Rightarrow \ddot{\underline{r}} = \ddot{\underline{r}}_0 - \ddot{\underline{R}}(t)$$

↳ in an inertial frame, $\ddot{\underline{R}}(t) = 0$ (i.e. constant velocity) so the equation of motion is the same.

↳ but for general $\underline{R}(t)$: $\boxed{m\ddot{\underline{r}} = \underline{F} - m\ddot{\underline{R}}}$. There is a fictitious force — e.g. in elevator going up, you feel force pushing down.

- Consider the case where S rotates with angular velocity $\underline{\omega}$

↳ the rate of change of unit vectors is given by $\dot{\underline{e}}_i = \underline{\omega} \times \hat{\underline{e}}_i$



↳ if the frames coincide at $t=0$

$$\underbrace{\underline{v}}_{\text{apparent velocity in } S} = \underbrace{\dot{\underline{r}}_0}_{\text{velocity in } S_0} - \underline{\omega} \times \underline{r}$$

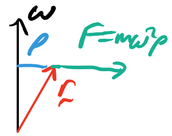
↳ the equation of motion is then

$$\boxed{\underbrace{m\underline{a}}_{\text{apparent}} = \underbrace{\underline{F}}_{\text{real}} - \underbrace{2m(\underline{\omega} \times \underline{v}) - m\,\underline{\omega} \times (\underline{\omega} \times \underline{r})}_{\text{fictitious}}}$$

- $-m\underline{\omega} \times (\underline{\omega} \times \underline{r})$ is the centrifugal force. i.e a constant force in the lab frame is required for rest in S

↳ $-m\underline{\omega} \times (\underline{\omega} \times \underline{r}) = m\omega^2(\underline{r} - (\underline{r}\cdot\hat{\underline{\omega}})\hat{\underline{\omega}})$

$$\Rightarrow F = m\omega^2\rho \text{ outwards}$$



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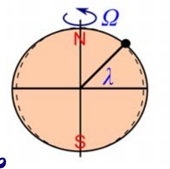
↳ centrifugal force explains the Earth's equatorial bulge. The rock deforms until it provides equal force in the space frame to cancel the centrifugal force.

- $-2m(\underset{\sim}{\omega} \times \underset{\sim}{v})$ is the Coriolis force, a 'swirl' which appears when moving within a rotating frame
  ↳ the Coriolis force on the Earth's surface is $F = 2m\Omega v \sin\lambda$, and points to the right when in the Northern hemisphere.



  ↳ for a falling body, $F = 2m\Omega v \cos\lambda$

- The motion of rotating frames can also be derived using an operator: $\boxed{\left[\frac{d}{dt}\right]_{S_0} = \left[\frac{d}{dt}\right]_S + \underset{\sim}{\omega} \times}$

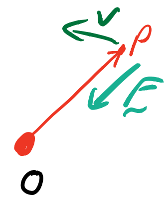
$$\therefore \left[\frac{d^2 \underset{\sim}{r_0}}{dt^2}\right]_{S_0} = \left(\left[\frac{d}{dt}\right]_S + \underset{\sim}{\omega} \times\right)\left(\left[\frac{d\underset{\sim}{r}}{dt}\right]_S + \underset{\sim}{\omega} \times \underset{\sim}{r}\right)$$

↳ this allows us to analyse the most general case, where an observer moves on a path $\underset{\sim}{R}(t)$ while using a rotating frame with changing $\underset{\sim}{\omega}(t)$
↳ the operator acts on $\underset{\sim}{\omega}$ too, leading to an additional fictitious force – the Euler force

$$m\underset{\sim}{a} = \underset{\sim}{F} - 2m(\underset{\sim}{\omega} \times \underset{\sim}{v}) - m\underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}) - m\dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}$$

Orbits

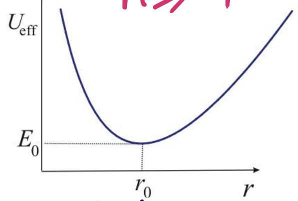
- Consider a particle moving in a central force field.
  ↳ the potential yields a purely radial force: $\underset{\sim}{F} = -\nabla V = -\frac{dV}{dr}\hat{\underset{\sim}{e}}_r$



  ↳ because the force exerts no couple, angular momentum is conserved
  $$\boxed{J = mr^2\dot{\phi} = \text{const}}$$
  ↳ thus motion is confined to a plane enclosing $\underset{\sim}{v}, \underset{\sim}{r}$.
  ↳ total energy is conserved:
  $$E = U(r) + \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) = \tfrac{1}{2}m\dot{r}^2 + U(r) + \frac{J^2}{2mr^2}$$
  ↳ we thus define the effective potential to include the angular velocity's contribution: $\boxed{U_{eff}(r) \equiv U(r) + \frac{J^2}{2mr^2}}$

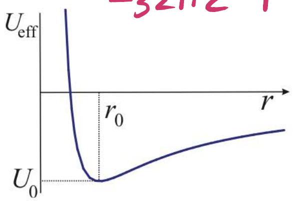
- Consider some attractive force $F = -Ar^n$, $A > 0$
  ↳ $U_{eff}(r) = \frac{Ar^{n+1}}{n+1} + \frac{J^2}{2mr^2}$ (unless $n = -1$)
  ↳ orbits correspond to equilibrium points $\left.\frac{dU_{eff}}{dr}\right|_{r=r_0}$

$n \geq -1$



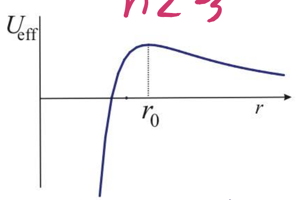
↳ stable at r_0
↳ all orbits bound

$-3 < n < -1$



↳ stable at r_0
↳ unbound for $E > 0$

$n < -3$



↳ unstable orbit
↳ $r \to 0$ or $r \to \infty$

- **Nearly-circular orbits** can be treated as oscillations about r_0. We can approximate $U_{eff}(r)$ as locally quadratic with a Taylor expansion about $r=r_0$, with $U'_{eff}(r_0)=0$ by definition. Alternatively, use $\dot{E}=0$:

$$\frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2 + U_{eff}\right) = \dot{r}\left(m\ddot{r} + \frac{dU_{eff}}{dr}\right) = 0$$

$$\frac{dU_{eff}}{dr} = Ar^n - \frac{J^2}{mr^3} \quad \text{but} \quad A = \frac{J^2}{m r_0^{n+3}}$$

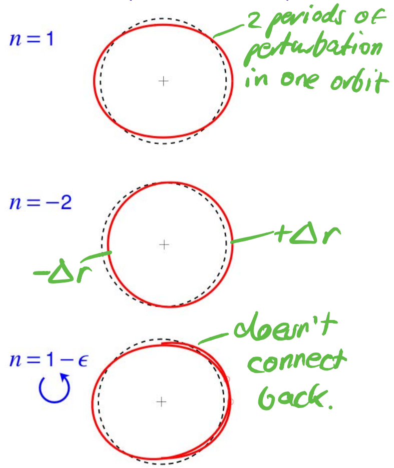
$$\therefore\ m\ddot{r} + \frac{(n+3)J^2}{m r_0^4}(r-r_0) = 0$$

this is the radial deviation from the circular path

↳ i.e. SHM with $\omega_p = \sqrt{n+3}\,\frac{J}{m r_0^2}$

↳ we can compare this to the angular freq of orbit $\Rightarrow \boxed{\omega_p = \sqrt{n+3}\,\omega_c}$

- The relationship between ω_p and ω_c determines the orbit:
  - ↳ $n=1$ (SHM) $\Rightarrow \omega_p = 2\omega_c$, i.e. ellipse centred at origin
    - ↳ SHM is separable in Cartesian and spherical coordinates
  - ↳ $n=-2$ (inverse square) $\Rightarrow \omega_p = \omega_c$, i.e. ellipse with focus at origin
  - ↳ $n=1-\epsilon$ leads to near-elliptical orbit that precesses

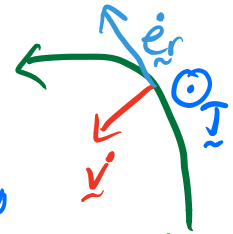


Inverse-square orbits

- Consider a force law $F = -A/r^2$, where $A = GMm$ for gravity.
- This force law implies **Kepler's laws**:

> K1. Planetary orbits are ellipses with the sun at one focus
>
> K2. The line joining a planet to the sun sweeps equal areas in equal times
> ↳ i.e. conservation of angular momentum
>
> K3. $T^2 \propto a^3$, where a is the semimajor axis

- For an orbit, $\underset{\sim}{J}, \underset{\sim}{v}, \dot{\underset{\sim}{e}}_r$ are mutually perpendicular (since acceleration is central)



$$\underset{\sim}{J} = mr^2\dot{\phi}\,\hat{\underset{\sim}{z}} \qquad \dot{\underset{\sim}{v}} = -\frac{A}{mr^2}\,\hat{\underset{\sim}{e}}_r \qquad \dot{\underset{\sim}{e}}_r = \dot{\phi}\,\hat{\underset{\sim}{e}}_\phi$$

$$\Rightarrow \underset{\sim}{J} \times \dot{\underset{\sim}{v}} = -A\,\dot{\underset{\sim}{e}}_r$$

↳ $\underset{\sim}{J}$ is constant so we integrate: $\underset{\sim}{J}\times\underset{\sim}{v} + A(\hat{\underset{\sim}{e}}_r + \underset{\sim}{e}) = 0$ (integration constant)

↳ dot both sides with $\underset{\sim}{r}$: $\underbrace{\underset{\sim}{J}\times\underset{\sim}{v}\cdot\underset{\sim}{r}}_{= \underset{\sim}{J}\cdot(\underset{\sim}{v}\times\underset{\sim}{r}) = -J^2/m} + A(r + \underset{\sim}{e}\cdot\underset{\sim}{r}) = 0$

$$\therefore\ r(1 + \underset{\sim}{e}\cdot\hat{\underset{\sim}{e}}_r) = \frac{J^2}{mA} \quad \Rightarrow \quad \boxed{r = \frac{r_0}{1+e\cos\phi}}$$

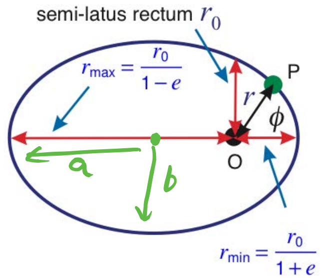
↳ this is the polar equation of an ellipse (K1) with $r_0 = \frac{J^2}{mA}$ and a focus at $r=0$.

- We can convert to Cartesians with

  $x = r\cos\phi$, $r = r_0 - ex$

  ↳ this gives the semimajor/minor axes

  $$a = \frac{r_0}{1-e^2} \qquad b = \frac{r_0}{\sqrt{1-e^2}}$$



  ↳ the periapsis and apoapsis depend on the semimajor axis and eccentricity

  $r_{max} = a(1+e)$ $\quad r_{min} = a(1-e)$ $\quad \Rightarrow 2a = r_{max} + r_{min}$

- The area of an ellipse is $\pi ab = \frac{\pi r_0^2}{(1-e^2)^{3/2}}$

  ↳ rate of sweeping is $\frac{1}{2} r^2 \dot\phi = \frac{J}{2m}$

  ↳ hence the period is $T = \frac{\pi r_0^2}{(1-e^2)^{3/2}} \Big/ \frac{J}{2m} = 2\pi\sqrt{\frac{ma^3}{A}}$

  this shows K3 →

- The energy of the orbit is given by $E = \frac{1}{2}mv^2 - \frac{A}{r}$

  $A\underline{e} = -(\underline{J}\times\underline{v} + A\hat{\underline{e}}_r)$ take dot product with itself

  $$\Rightarrow A^2e^2 = J^2v^2 + 2\underbrace{(\underline{J}\times\underline{v}\cdot\hat{\underline{e}}_r)}_{= \underline{J}\cdot\underline{v}\times\hat{\underline{e}}_r = -Jv = -J^2/mr} A + A^2$$

  $$\therefore A^2(e^2-1) = J^2\left(v^2 - \frac{2A}{mr}\right) \Rightarrow E = \frac{A(e^2-1)}{2r_0}$$

  ↳ hence the energy is related to the major axis: $\boxed{E = -\frac{A}{2a}}$

  (independent of eccentricity)

  ↳ angular momentum depends on r_0: $\boxed{J^2 = Amr_0}$

- Kepler's laws can instead be derived by considering energy:

  $$E = \frac{1}{2}m\dot r^2 + \frac{J^2}{2mr^2} - \frac{A}{r}$$

  ↳ sub $u = 1/r$ to simplify algebra, then complete the square.

Unbound orbits

- The eccentricity of the orbit can be written in terms of E and J: $\quad e^2 = 1 + \frac{2EJ}{mA^2}$

  ↳ $0 \le e < 1$: the orbit is bound and E is negative

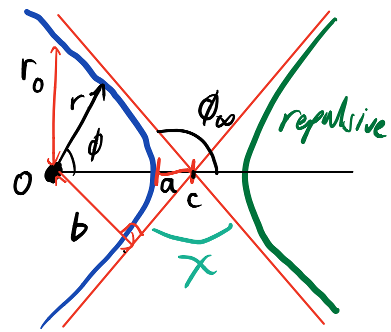
  ↳ $e = 1$: unbound parabolic orbit, $E = 0$

  ↳ $e > 1$: unbound hyperbolic orbit, E positive.

- For a parabolic orbit, the focal length is $f \equiv r_{min} = \frac{1}{2}r_0$

  $$r = \frac{r_0}{1+\cos\phi} \Rightarrow r = r_0 - x \Rightarrow y^2 = 4f(f-x)$$

- For hyperbolic orbits, $e > 1 \Rightarrow a < 0$ (semimajor), but all previous formulae are valid.



  ↳ the impact parameter b and velocity at infinity v_∞ determine E and J:

  $$J = mbv_\infty \qquad E = \frac{1}{2}mv_\infty^2$$

  ↳ the total angle of deflection is $\chi = 2\phi_\infty - \pi$, with $\cos\phi_\infty = -1/e$

  $$\Rightarrow \boxed{|\tan\phi_\infty| = \frac{mv_\infty^2 b}{A}}$$

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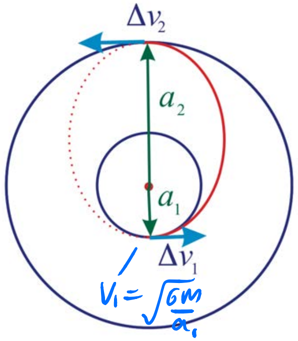
- For a repulsive inverse-square force (e.g. Rutherford scattering), we use the other branch of the hyperbola
  - ↳ distance of closest approach is $\boxed{a(1+e)}$
  - ↳ this can instead be derived by integrating the force.

Changing an orbit

- The most efficient way to move between two orbits is the Hohmann Transfer orbit
  - ↳ the change in energy to move into the transfer orbit ← elliptical:

$$E_t = -\frac{GMm}{a_1+a_2} = -\frac{GMm}{a_1} + \frac{1}{2}mV_t^2$$



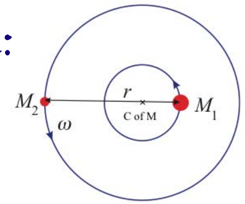
$$V_1 = \sqrt{\frac{GM}{a_1}}$$

  - ↳ then $\Delta V_1 = V_t - V_1$
  - ↳ likewise, there will be another ΔV_2 to move from the transfer orbit into the larger circular orbit.

- If there is another planet, a gravitational slingshot can be used to change an orbit (normally to increase speed).
  - ↳ e.g if there is a fast planet, the probe can enter an unbound orbit around the planet
  - ↳ convert GPE → KE

The N-body problem

- For a constant external potential, the two-body problem can be solved exactly.
  - ↳ each orbit is an ellipse in a common plane with the centre of mass at one focus.
  - ↳ balancing gravity and the centrifugal force:

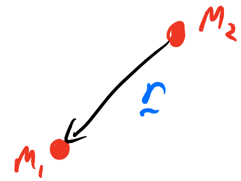


$$\frac{GM_1M_2}{r^2} = M_1\omega^2\frac{M_2 r}{M_1+M_2}$$

  - ↳ if we use the reduced mass, this simplifies:

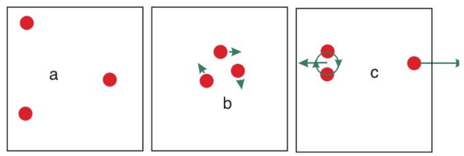
$$\mu = \frac{M_1M_2}{M_1+M_2}, \qquad \mu r\omega^2 = \frac{GM_1M_2}{r^2}$$

  - ↳ other relations can then be written directly in terms of the separation vector $\underset{\sim}{r}$



$$\left.\begin{aligned} T &= \tfrac{1}{2}\mu\dot{\underset{\sim}{r}}^2 \\ \underset{\sim}{J} &= \mu\,\underset{\sim}{r}\times\dot{\underset{\sim}{r}} \\ \underset{\sim}{F} &= \mu\,\ddot{\underset{\sim}{r}} \end{aligned}\right\}\ \text{reduced to a 1 body problem in CoM frame.}$$

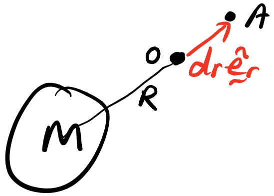
- However, for $N \geq 3$, the N-body problem does not generally have an exact solution unless interactions are simple harmonic.
  - ↳ generally, 3-body interactions result in a close binary forming, which may release enough KE for one body to escape.



Tidal forces

- The gravitational potential $\phi(\underline{r})$ is only defined w.r.t some constant reference. $\underline{g}(\underline{r}) = -\nabla\phi$
- However, for a distant source, all objects are uniformly accelerating towards it so there is no measurable effect.
  - ↳ the only thing that can be measured is the tidal field $\boxed{\underline{T}(\underline{a}) = (\underline{a}\cdot\nabla)\underline{g}}$, which describes how $\underline{g}$ varies between points $\underline{r}_0$ and $\underline{r}_0 + \underline{a}$

  - ↳ for a small radial change $dr\hat{\underline{e}}_r$:

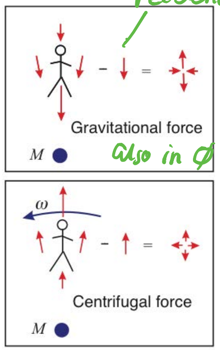


$$\Delta g = -\frac{GM}{(R+dr)^2} - \left(-\frac{GM}{R^2}\right) \Rightarrow \underline{T}(\hat{\underline{e}}_r) = \frac{2GM}{R^3}\hat{\underline{e}}_r$$

  - ↳ along $\hat{\underline{e}}_\theta$, $|g|$ doesn't change

$$\therefore |g|\,d\theta\,\hat{\underline{e}}_\theta = -\frac{GM}{R^2}d\theta\,\hat{\underline{e}}_\theta \Rightarrow \underline{T}(\hat{\underline{e}}_\theta) = -\frac{GM}{R^3}\hat{\underline{e}}_\theta$$

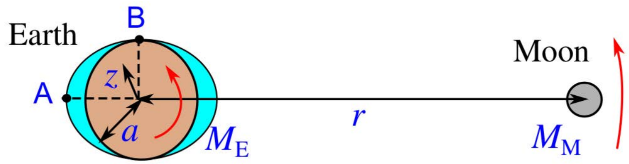
  - ↳ same for $\hat{\underline{e}}_\phi$: $\underline{T}(\hat{\underline{e}}_\phi) = -\frac{GM}{R^3}\hat{\underline{e}}_\phi$



- Hence an object in a gravitational field experiences radial stretching and lateral squeezing
  - ↳ if the object is also orbiting, then there is another contribution from the centrifugal force ↖ not in ϕ
  - ↳ the net result is:

$3GM/R^3$ stretch	$-GM/R^3$	none
(Radial)	($\perp$ to orbital plane)	(in orbital plane)

- On Earth, water moves in response to the moon's gravity.



← moon 2x stronger than sun.

  - ↳ at a distance z from the centre, the difference in field causes a radial stretching of $2GM_m z/r^3$ and tangential compression of $-GM_m z/r^3$ (no centrifugal contrib).
  - ↳ integrating both with $\int_0^a dz$, the tidal potential difference as a result of the moon is $\phi_{tide} = \frac{3GM_m a^2}{2r^3}$
  - ↳ From the Earth, $\phi_{tide} = gh$ where g is assumed to be constant: $g = GM_e/a^2$.
  - ↳ equating these gives the height of the tides.
- The Earth rotates w.r.t the two bulges of water, hence there are two tides a day
  - ↳ the tidal field from the sun complicates things
  - ↳ Friction from the water slows down the Earth. The moon recedes to conserve angular momentum.

Robert Andrew Martin

Rigid Body Dynamics

- A rigid-body is a many-particle system in which all inter-particle distances are fixed
- For a general rigid body:

vector triple product.

$$\underset{\sim}{J} = \sum \underset{\sim}{r} \times \underset{\sim}{p} = \sum \underset{\sim}{r} \times m(\underset{\sim}{\omega} \times \underset{\sim}{r}) = \sum m r^2 \underset{\sim}{\omega} - \sum m \underset{\sim}{r}(\underset{\sim}{\omega} \cdot \underset{\sim}{r})$$

↳ hence in general, $\underset{\sim}{J}$ is not parallel to $\underset{\sim}{\omega}$. They are related by the inertia tensor $\underline{\underline{I}}$ ← a matrix

↳ expanding in Cartesians gives:

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} \sum m(y^2+z^2) & -\sum mxy & -\sum mxz \\ -\sum mxy & \sum m(x^2+z^2) & -\sum myz \\ -\sum mxz & -\sum myz & \sum m(x^2+y^2) \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

- The kinetic energy of a rotating rigid body is

$$T = \sum \tfrac{1}{2} m(\underset{\sim}{\omega} \times \underset{\sim}{r}) \cdot (\underset{\sim}{\omega} \times \underset{\sim}{r}) = \tfrac{1}{2} \underset{\sim}{\omega} \cdot \underline{\underline{I}} \underset{\sim}{\omega} \Rightarrow \boxed{T = \tfrac{1}{2} \underset{\sim}{\omega} \cdot \underset{\sim}{J}}$$

- Because $\underline{\underline{I}}$ is symmetric and real, it has 3 real eigenvalues $\{I_1, I_2, I_3\}$ and orthogonal eigenvectors.
  - ↳ $\{I_1, I_2, I_3\}$ are the principle moments of inertia
  - ↳ $\{\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_3\}$ are the principle axes
  - ↳ in the eigenbasis:

$$\boxed{\underline{\underline{I}}' = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \qquad \underset{\sim}{J} = \begin{pmatrix} I_1 \omega_1 \\ I_2 \omega_2 \\ I_3 \omega_3 \end{pmatrix}}$$

  - ↳ $\underline{\underline{I}} \underset{\sim}{\omega}_K = I_K \underset{\sim}{\omega}_K$ (no sum) defines $\underset{\sim}{\omega}_K$ as a principal axis.
- The KE in the eigenbasis is $T = \frac{1}{2}(I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2)$

↳ the surface of constant KE in ω-space is an ellipsoid

↳ this inertia ellipsoid is fixed to the body, and has axes of length $\propto 1/\sqrt{I_i}$

↳ $\underset{\sim}{J}$ is perpendicular to the surface of the inertia ellipsoid, i.e $\boxed{\nabla_\omega T = \underset{\sim}{J}}$

- For an object to rotate smoothly on an axis, it must be:
  - statically balanced i.e axis passes through COM
  - dynamically balanced i.e axis is a principal axis.
- The character of the principal axes depends on symmetry:
  - spherical tops (e.g sphere, cube) are balanced around the COM. I is scalar and is the same about any axis through COM.
  - symmetrical tops have $I_1 = I_2 \neq I_3$. $\hat{\underset{\sim}{e}}_3$ is unique and normal to the plane containing $\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2$.
  - asymmetrical tops have $I_1 \neq I_2 \neq I_3$
- No one I_i can be larger than the sum of the others. The limiting case is a lamina, which results in the perpendicular axes theorem: $I_1 + I_2 = I_3$
- The parallel axes theorem states that I about an axis parallel to the COM, separated by a, is:

$$I = I_0 + Ma^2$$

↳ in a general basis, we instead need $\underline{\underline{I}} = \underline{\underline{I}}_0 + \underline{\underline{I}}_R$ where $\underline{\underline{I}}_R$ is the inertia tensor of a point mass at the CM about the origin, and $\underline{\underline{I}}_0$ is the inertia tensor about the COM.

Free precession and Euler's equations

- Euler's equations consider the change in $\underset{\sim}{J}$ in the body frame S, which rotates with respect to an inertial frame S_0.
  - ↳ NII : $\left[\frac{d\underset{\sim}{J}}{dt}\right]_{S_0} = \underset{\sim}{G}$
  - ↳ coordinate transform: $\left[\frac{d}{dt}\right]_{S_0} = \left[\frac{d}{dt}\right]_S + \underset{\sim}{\omega}\times$
  - ⇒ equation of motion: $\boxed{\underset{\sim}{G} = \left[\frac{d\underset{\sim}{J}}{dt}\right]_S + \underset{\sim}{\omega}\times\underset{\sim}{J}}$
  - ↳ this can be expanded easily since we are in the eigenbasis.

$$\boxed{G_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2}$$ and cyclic perms

- For a symmetric top, $I_1 = I_2 = I \neq I_3$. Euler's equations are:

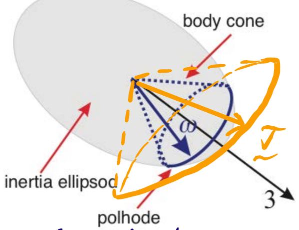
$$\left.\begin{aligned} I\dot{\omega}_1 &= (I - I_3)\omega_2\omega_3 \\ I\dot{\omega}_2 &= (I_3 - I)\omega_1\omega_3 \\ I_3\dot{\omega}_3 &= 0 \end{aligned}\right\}$$ let $\boxed{\Omega_b \equiv \frac{I - I_3}{I}\omega_3}$ be the body frequency

$$\Rightarrow \dot{\omega}_1 + \Omega_b\omega_2 = 0, \quad \dot{\omega}_2 - \Omega_b\omega_1 = 0$$

↳ solving the coupled ODEs shows that $\underset{\sim}{\omega}$ precesses around the 3-axis (tracing a cone) in the body frame.

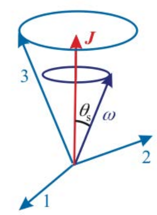
↳ $\underset{\sim}{J} = \underset{\approx}{I}\underset{\sim}{\omega}$, so $\underset{\sim}{J}$ also traces a cone.

↳ the sign of Ω_b determines whether the inertia ellipsoid is oblate or prolate



(prolate top)

- In the space frame, we require $\underset{\sim}{J}$ to be constant (no external torques). The 3-axis and $\underset{\sim}{\omega}$ rotate around $\underset{\sim}{J}$ at the space frequency.



$$\underset{\sim}{\omega} = (\omega_1\hat{\underset{\sim}{e}}_1 + \omega_2\hat{\underset{\sim}{e}}_2) + \omega_3\hat{\underset{\sim}{e}}_3$$

$$\underset{\sim}{J} = I(\omega_1\hat{\underset{\sim}{e}}_1 + \omega_2\hat{\underset{\sim}{e}}_2) + I_3\omega_3\hat{\underset{\sim}{e}}_3$$

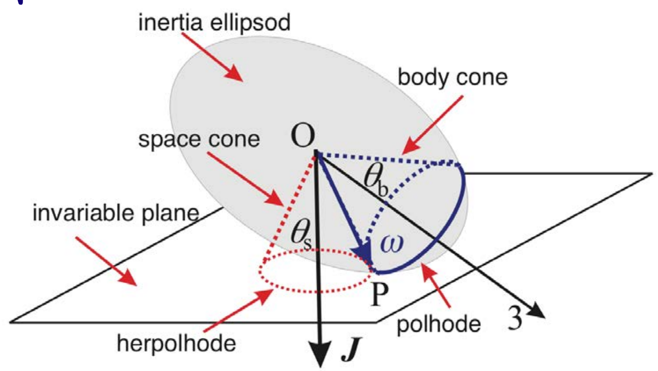
↳ we can eliminate and write $\underset{\sim}{\omega} = \frac{\underset{\sim}{J}}{I} - \Omega_b\hat{\underset{\sim}{e}}_3$

↳ linear relationship so $\underset{\sim}{\omega}, \underset{\sim}{J}, \hat{\underset{\sim}{e}}_3$ are coplanar

$$\frac{d}{dt}\left[\hat{\underset{\sim}{e}}_3\right] = \underset{\sim}{\omega}\times\hat{\underset{\sim}{e}}_3 = \left(\frac{J}{I}\hat{\underset{\sim}{J}} - \Omega\hat{\underset{\sim}{e}}_3\right)\times\hat{\underset{\sim}{e}}_3 = \left(\frac{J}{I}\hat{\underset{\sim}{J}}\right)\times\hat{\underset{\sim}{e}}_3$$

↳ this means that $\hat{\underset{\sim}{e}}_3$ (and thus $\underset{\sim}{\omega}$) are rotating around $\hat{\underset{\sim}{J}}$ with space frequency $\boxed{\Omega_s = J/I}$

- Poinsot's construction is a geometric treatment relating the body/space cones:



↳ constant $\underset{\sim}{J}$ and $T = \frac{1}{2}\underset{\sim}{\omega}\cdot\underset{\sim}{J}$, so component of $\underset{\sim}{\omega}$ along $\underset{\sim}{J}$ must be constant. So tip of $\underset{\sim}{\omega}$ stays on plane.

↳ the contact point P is instantaneously at rest, so the ellipsoid rolls – i.e. cones rotate around each other

↳ can relate frequencies using $\Omega_b\sin\theta_b = \Omega_s\sin\theta_s$

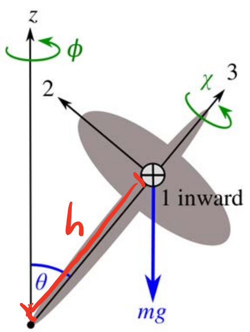
Robert Andrew Martin

- A triaxial body has 3 different principal moments $I_1 < I_2 < I_3$
  - ↳ to analyse, use conservation laws $\underset{\sim}{J} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$
    $T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$
  - ↳ rotation around 1-axis or 3-axis is stable because $\underset{\sim}{\omega}$ cannot be changed at constant $\underset{\sim}{J}$ without changing T.
  - ↳ but rotation about the 2-axis is unstable.

- The Major axis theorem states that any freely-rotating body that is not perfectly rigid will lose energy until it aligns with its major axis:
  - ↳ because of centrifugal forces, a non-rigid body deforms and thus loses energy
  - ↳ $\underset{\sim}{J}$ is fixed, so the resulting rotation minimises energy for constant J by aligning $\underset{\sim}{J}$ with the largest I.

Gyroscopes and Lagrange's approach

- Consider a heavy symmetric top pivoted at base.



- We define the Euler angles (θ, ϕ, χ):

$$\underset{\sim}{\omega} = \dot{\phi}\,\hat{\underset{\sim}{e}}_z + \dot{\theta}\,\hat{\underset{\sim}{e}}_1 + \dot{\chi}\,\hat{\underset{\sim}{e}}_3$$
$$= \dot{\theta}\,\hat{\underset{\sim}{e}}_1 + \dot{\phi}\sin\theta\,\hat{\underset{\sim}{e}}_2 + (\dot{\chi} + \dot{\phi}\cos\theta)\,\hat{\underset{\sim}{e}}_3$$

- Gravity exerts some torque $G_1 = mg\sin\theta$:
  - ↳ $J_3 = I_3(\dot{\chi} + \dot{\phi}\cos\theta)$ is constant $\because G_3 = 0$
  - ↳ $J_z = J_3\cos\theta + J_2\sin\theta = J_3\cos\theta + I\dot{\phi}\sin^2\theta$ is const $\because G_z = 0$

- Hence $\dot{\phi}$ and $\dot{\chi}$ can be expressed in terms of the constants J_3, J_z as well as θ.

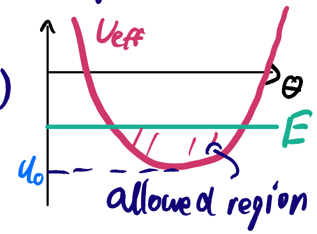
$$\dot{\phi} = \frac{J_z - J_3\cos\theta}{I\sin^2\theta} \qquad \dot{\chi} = \frac{J_3}{I_3} + \frac{J_3\cos^2\theta - J_z\cos\theta}{I\sin^2\theta}$$

$I \equiv I_1 = I_2$ is taken about the point of support

- ↳ $\theta(t)$ can be found from conservation of energy:

$$E = \frac{1}{2}I\dot{\theta}^2 + \underbrace{\frac{(J_z - J_3\cos\theta)^2}{2I\sin^2\theta} + \frac{J_3^2}{2I_3} + mgh\cos\theta}_{=\,U_{eff}(\theta)} = \text{const}$$

- ↳ in principle, this gives θ and thus ϕ, χ. However, it is easier to reason in terms of the effective potential.



- If the energy is $\geqslant$ the min U_{eff} ($\equiv U_0$) there is some allowed region of θ $(\dot{\theta}^2 \geqslant 0)$
- The value of E determines what kind of precession occurs.
- If $E = U_0$ there is one stable value of θ so we have steady precession
  - ↳ for $\theta = \theta_0$ cons angular momentum $\Rightarrow$ $\dot{\phi}, \dot{\chi} = \text{const}$
  - ↳ θ_0 can be found with $U_{eff}'(\theta_0) = 0$, leading to:

$$\dot{\phi} = \frac{J_3 \pm \sqrt{J_3^2 - 4I_1 mgh\cos\theta}}{2I_1\cos\theta} \quad \Leftarrow \cos\theta > 0$$

  - ↳ hence steady precession requires the gyroscope flywheel to be rotating sufficiently fast such that $J_3^2 \geqslant 4I_1 mgh\cos\theta$

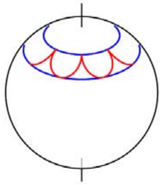
Robert Andrew Martin

↳ in the gyroscopic limit $J_3^2 >> mghI$, we can Taylor expand to find two solutions.

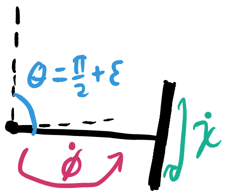
↳ slow precession: $\dot{\phi} \approx mgh/J_3$

↳ fast precession: $\dot{\phi} \approx J_3/(I\cos\theta)$ ← i.e neglect couple

- If $E > U_0$, we can Taylor expand the potential about the minimum; $\theta(t)$ undergoes SHM:
  - ↳ hence $\dot{\phi}$ and $\dot{\chi}$ also oscillate
  - ↳ the resulting motion is called nutation, and is generally quite complex.



- A simple case of nutation is for a horizontal gyroscope:
  - ↳ expand U_{eff} about $\pi/2$ in the gyroscopic limit:
    $U_{eff}(\theta) \approx \text{const} + \frac{1}{2}\frac{J_3^2}{I}\varepsilon^2$
  - ↳ i.e SHM with frequency $\Omega_s = J_3/I$



Lagrangian Dynamics

- Hamilton's principle states that a system follows a path that extremises the action functional $S = \int_{t_0}^{t_1} \mathcal{L}(q_i, \dot{q}_i, t)\,dt$, where $\mathcal{L}$ is the Lagrangian, such that $\mathcal{L} = T - V$.
- For fixed endpoints, $\delta S = 0$ implies the Euler-Lagrange equations:

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \frac{\partial \mathcal{L}}{\partial q_i}}, \quad \forall i$$

- The terms $\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \equiv p_i$ are conjugate momenta
  - ↳ if the Lagrangian is independent of a coordinate q_i then the conjugate momentum p_i is constant.
  - ↳ symmetries are closely related to conservation laws.
- The Lagrangian does not define energy, so we form the Hamiltonian:

$$\boxed{H(q_i, p_i, t) \equiv \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t)}$$

$$\Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \boxed{\frac{dH}{dt} = -\frac{\partial \mathcal{L}}{\partial t}}$$

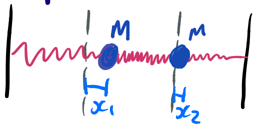
- If the Lagrangian is time-independent, the Hamiltonian is conserved.
- e.g SHM: $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
  - ↳ $\frac{\partial \mathcal{L}}{\partial t} = 0 \Rightarrow E$ conserved
- e.g Orbits: $\mathcal{L} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$
  - ↳ $\frac{\partial \mathcal{L}}{\partial \phi} = 0 \Rightarrow p_\phi = J = mr^2\dot{\phi}$ conserved
- e.g Symmetric top: $\mathcal{L} = \frac{1}{2}I(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + \frac{1}{2}J_3(\dot{\chi} + \dot{\phi}\cos\theta)^2 - mgh\cos\theta$
  - ↳ $p_\phi = J_z$ and $p_\chi = J_3$ are conserved.

Robert Andrew Martin

Normal Modes

- In general, small free displacements of a system about equilibrium lead to linear equations.
- In a normal mode, every element of the system oscillates at a single frequency. But a given system may have multiple normal modes (each with a different freq).

- Consider a two-mass system with three ideal springs. The equations of motion (which can be found from Hamilton's principle) can be written in matrix form:



$$\begin{pmatrix} m\ddot{x}_1 \\ m\ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

↳ we use the trial solution $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} e^{i\omega t}$

↳ this results in homogeneous linear equations for the constants X_1, X_2:

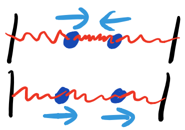
$$\begin{pmatrix} 2k-m\omega^2 & -k \\ -k & 2k-m\omega^2 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

↳ nontrivial solutions iff determinant is zero

$\Rightarrow \omega^2 = 3k/m$ or $\omega^2 = k/m$

↳ either $X_1 + X_2 = 0$ with mode $\propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

or $X_1 - X_2 = 0$ with mode $\propto \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



↳ in this case we could have guessed the normal modes by symmetry then found freq with $\omega^2 = \frac{\text{restoring force constant}}{\text{mass}}$

- Consider a general system specified by N generalised coordinates $\{q_i\}$. Suppose that the equilibrium position is $q_i = 0, \forall i$. The KE is then $T = \frac{1}{2}\sum_k m_k |\dot{r}_k|^2$

↳ this can be written as a quadratic function of the coordinates: $T = \frac{1}{2}\dot{\underline{q}}^T \underline{\underline{M}}\, \dot{\underline{q}}$ where $\underline{\underline{M}}$ is the mass matrix

↳ likewise, we can write $U = \frac{1}{2}\underline{q}^T \underline{\underline{K}}\, \underline{q}$. by construction, $\underline{\underline{M}}$ and $\underline{\underline{K}}$ must be symmetric

↳ $\frac{d}{dt}(T+U) = 0 \Rightarrow \boxed{\underline{\underline{M}} \cdot \ddot{\underline{q}} + \underline{\underline{K}} \cdot \underline{q} = 0}$

↳ we then proceed the same way as before.

- The normal mode theorem states that for a system with N coordinates and quadratic KE/PE, we can find N 'orthogonal' oscillatory modes.

↳ $(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}) \cdot \underset{\sim}{X}$ is not a true eigenvalue equation, so modes X are not technically orthogonal.

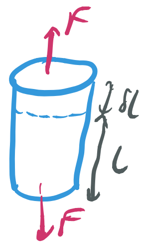
↳ however, $\underset{\sim}{X}_i^T \cdot \underline{\underline{M}} \cdot \underset{\sim}{X}_j = 0$ for $i \neq j$

- If all ω^2s are positive, the system is stable. Negative ω^2s correspond to exponentially growing modes.
- Degeneracy is when normal mode frequencies are equal ← might just be accidental
- ☆ General free oscillation is a superposition of normal modes.

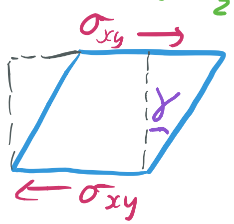
e.g $$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathrm{Re}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}(At+B) + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{i\omega_1 t} + \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{i\omega_2 t} \right\}$$

Robert Andrew Martin

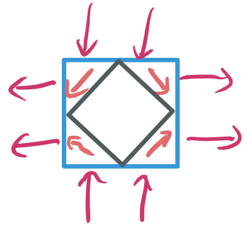
Elasticity



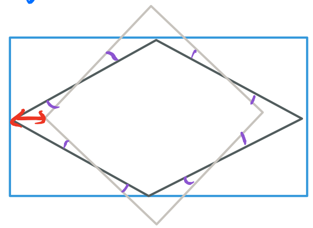
- Strain is the relative change in a dimension when a stress (force / area) is applied.
- In the elastic region, they are directly proportional.
  $\sigma = E\varepsilon$, where E is **Young's modulus**
- Usually, a strain in one dimension corresponds to a compression in orthogonal directions. The **Poisson ratio** ν encodes this.
  ↳ for a unit cube, stress along the x-axis causes strains
  $$E(\varepsilon_x, \varepsilon_y, \varepsilon_z) = \sigma_x(1, -\nu, -\nu)$$
  ↳ likewise for σ_y, σ_z.
- For an isotropic medium under uniform pressure:
  $\sigma_x = \sigma_y = \sigma_z = -P \Rightarrow \varepsilon_x = \varepsilon_y = \varepsilon_z = -P(1-2\nu)$
  ↳ to first order, the change in volume of the cube is
  $$\delta V = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z) \approx 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z$$
  ↳ the **bulk modulus** B is the constant of proportionality between applied pressure and the decrease in volume
  $$P = -B\frac{\delta V}{V} \Rightarrow \boxed{B = \frac{E}{3(1-2\nu)}}$$
  ← $B > 0$ for stable medium $\Rightarrow \nu < \frac{1}{2}$
- If a stress is applied parallel to the surface, it is a **shear stress**, defined by a shear angle.



  ↳ must be symmetric for no net couple $\Rightarrow \sigma_{xy} = \sigma_{yx}$



  ↳ can be produced by a combination of tensile and compressive stress $\sigma_x = -\sigma_y$
  ↳ the shear force is then $\sigma/\sqrt{2}$ on a side length $1/\sqrt{2}$, so the shear stress is σ
  ↳ the associated strain is $E\varepsilon_x = \sigma_x - \nu\sigma_y = \sigma_x(1+\nu)$
- The shear angle is the total angular change from the once-parallel sides. $\gamma \equiv 2\varepsilon_x$



  ↳ the **shear modulus** is then
  $$G = \frac{\sigma_{xy}}{\gamma} = \frac{E\varepsilon_x}{1+\nu}\Big/ 2\varepsilon_x \Rightarrow \boxed{G = \frac{E}{2(1+\nu)}}$$

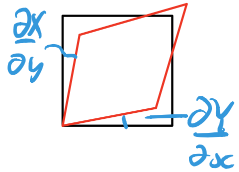
- Formally, stress is represented as the symmetric **stress tensor** $\underline{\sigma}$, where each element σ_{xy} is the force/area in the x direction transmitted along the y plane
  ↳ since it is symmetric, it can be diagonalised
  ↳ hence arbitrary stresses can be represented as principal components $(\sigma_1, \sigma_2, \sigma_3)$
  ↳ antisymmetric components represent a couple, so can be extracted and analysed separately.
- A strain can be thought of as a distortion that moves each point by a variable amount, i.e. $\underline{x} \to \underline{x} + \underline{\chi}(\underline{x})$
  ↳ two nearby points are moved by different amounts, where the difference is related to the gradient of $\underline{\chi}$

original x ——— $x+\Delta x$

distorted $x+X$ ——— $x+X+\Delta x\frac{\partial X}{\partial x}$

$\Rightarrow \varepsilon_{xx} = \frac{\partial X}{\partial x}$

- The shear angle in the xy plane is $\frac{\partial Y}{\partial x}+\frac{\partial X}{\partial y} \Rightarrow \varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}\left(\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}\right)$



- This can all be summarised in the symmetric strain tensor

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}, \quad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial X_i}{\partial x_j}+\frac{\partial X_j}{\partial x_i}\right)$$

↳ the distortion due to a strain is $\delta \underline{X} = \underline{\underline{\varepsilon}}\,\delta \underline{x}$

↳ can always be diagonalised into principal axes.

- In an isotropic medium, the principal axes are the same for both the stress and strain tensors.
- The relationship between stress and strain can then be found by solving $E(\varepsilon_x, \varepsilon_y, \varepsilon_z) = \sigma_x(1, -\nu, -\nu)$ and its cyclic permutations.

↳ $\sigma_1 = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_1 + \nu\varepsilon_2 + \nu\varepsilon_3\right]$

↳ this results in a part of stress proportional to strain and a pressure proportional to the change in volume, $\mathrm{Tr}\,\underline{\underline{\varepsilon}}$

$$\boxed{\underline{\underline{\sigma}} = \lambda\,\mathrm{Tr}\,\underline{\underline{\varepsilon}}\,\underline{\underline{I}} + 2G\underline{\underline{\varepsilon}}}$$

with $\lambda \equiv \frac{E\nu}{(1+\nu)(1-2\nu)} = B - \frac{2}{3}G$

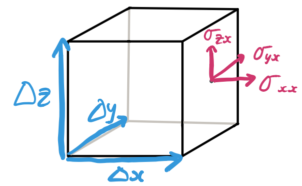
Stored energy

- $W = \frac{1}{2}kx^2 = \frac{1}{2}Fx$. Along the x face:

↳ distortion is $\Delta x(\varepsilon_{xx}, \varepsilon_{yx}, \varepsilon_{zx})$

↳ force is $\Delta y\Delta z(\sigma_{xx}, \sigma_{yx}, \sigma_{zx})$

$\Rightarrow W = \frac{1}{2}V(\varepsilon_{xx}\sigma_{xx} + \varepsilon_{yx}\sigma_{yx} + \varepsilon_{zx}\sigma_{zx})$



↳ we then need to add over all pairs

↳ in the principal axes, this simplifies to:

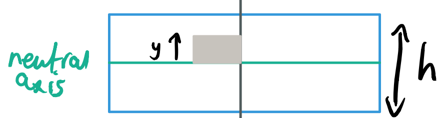
stored energy per unit volume → $U = \boxed{\frac{1}{2}(\sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 + \sigma_3\varepsilon_3)}$

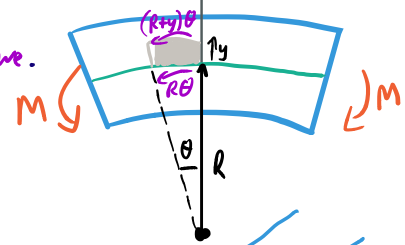
- We can use the expression for $\underline{\underline{\sigma}}$ in terms of $\underline{\underline{\varepsilon}}$ to find U in terms of $\underline{\underline{\varepsilon}}$:

$$U(\underline{\underline{\varepsilon}}) = \frac{1}{2}\left[\lambda(\mathrm{Tr}\,\underline{\underline{\varepsilon}})^2 + 2G\,\mathrm{Tr}(\underline{\underline{\varepsilon}}^2)\right]$$

Beam theory

- Consider a beam subject to pure bending (no shear).
- The top will be subject to tension and the bottom to compression, but there will be an undistorted neutral axis, from which we define the radius of curvature.



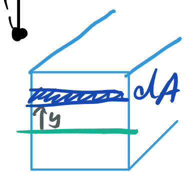


$$\varepsilon = \frac{(R+y)\theta - R\theta}{R\theta} = \frac{y}{R} \Rightarrow \boxed{\sigma = \frac{Ey}{R}}$$

- Hence the bending moment is:

$B = \int y \cdot \sigma\, dA = EI/R$, where

$\boxed{I \equiv \int y^2 dA}$ is the second moment of area.



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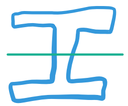
- To increase the beam rigidity we need to have more area away from the neutral axis.

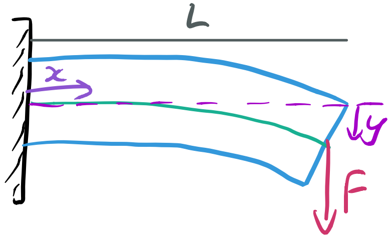


- For beams with two orthogonal principal axes, force and deflection will be parallel.

- In a cantilever beam, bending moment is a function of x: $B = F(L-x)$
  - ↳ for small deflections, the RoC can be approximated as $1/R \approx \frac{d^2y}{dx^2}$

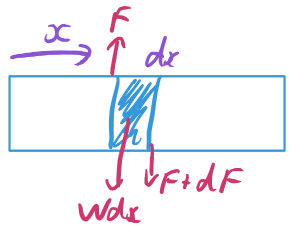
$$\therefore EI y'' = F(L-x)$$

$$\Rightarrow \quad y(x) = \frac{Fx^2}{6EI}(3L-x)$$

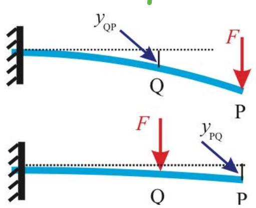


- For a general beam in equilibrium, we consider the load per unit length $W(x)$ and equate forces / moments.
  - ↳ $dF = W(x)dx$, $dB = Fdx$

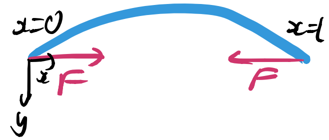
$$\therefore W = \frac{d^2B}{dx^2} = EI y'''' \quad \leftarrow \text{for small displacements}$$



- Calculation may be simplified by the reciprocity theorem – the deflection at Q due to load F at point P is the same as the deflection at P due to load F at Q. This can be shown by considering energy stored when loads at P, Q are added sequentially.



- An Euler strut is made by buckling a beam with force F.
  - ↳ the bending moment at x is $B = -Fy(x)$



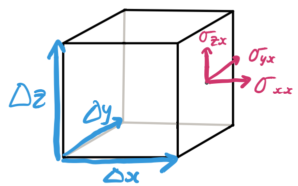
$$\Rightarrow \quad y'' + \frac{F}{EI}y = 0 \quad \Rightarrow \quad \sqrt{\frac{F}{EI}}\,L = \pi \quad \leftarrow \text{to fit sine to B.Cs}$$

- ↳ the Euler force is then $\boxed{F_E = \frac{\pi^2 EI}{L^2}}$
- ↳ for $F < F_E$, the beam is compressed but doesn't buckle
- ↳ for $F \geq F_E$, it will suddenly buckle.
- ↳ for a vertical cantilever of length $L/2$, the result is the same.

Dynamics of elastic media

- The net force in the x direction is

$$F_x = V \cdot \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} \right)$$



- ↳ hence the equation of motion is

$$\rho \frac{\partial^2 X_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j} \quad \text{i.e} \quad \boxed{\rho \frac{\partial^2 \underline{X}}{\partial t^2} = \nabla \cdot \underline{\underline{\sigma}}}$$

- ↳ using the stress-strain relation with $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial X_i}{\partial x_j} + \frac{\partial X_j}{\partial x_i}\right)$ results in the vector equation of motion

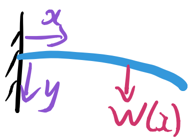
$$\boxed{\rho \frac{\partial^2 \underline{X}}{\partial t^2} = \left(B + \tfrac{1}{3}G\right)\nabla(\nabla \cdot \underline{X}) + G\nabla^2 \underline{X}}$$

- We can find wavelike solutions with $\underline{X} = (X_0, Y_0, Z_0)e^{i(\omega t - kx)}$

$$\begin{pmatrix} -\omega^2 X_0 \\ -\omega^2 Y_0 \\ -\omega^2 Z_0 \end{pmatrix} = \left(B + \tfrac{1}{3}G\right)\begin{pmatrix} -k^2 X_0 \\ 0 \\ 0 \end{pmatrix} + G\begin{pmatrix} -k^2 X_0 \\ -k^2 Y_0 \\ -k^2 Z_0 \end{pmatrix}$$

- ↳ for transverse disturbances (i.e. in y, z), the result is a S-wave (S for shear) with $\rho\omega^2 = G k^2$. This is nondispersive with $v_s^2 = G/\rho$
- ↳ for longitudinal disturbance, we have a P-wave (compression) with $V^2 = (B + 4/3\, G)/\rho$
- ↳ P-waves are thus faster than S-waves.
- Boundary conditions may be:
  - ↳ free → no normal stress $(\underline{\sigma} \cdot \underline{n}) \cdot d\underline{s} = 0$
  - ↳ fixed → no distortion $\underline{n} \cdot \underline{X} = 0$
- The energy flow in a wave is $P = -\dot{y}(Ty')$ ← velocity × transverse force
  - ↳ in general, this becomes $\boxed{P = -\underline{\underline{\tau}} \cdot \dot{\underline{X}}}$

Normal modes of an elastic bar



- For a cantilever, the force required to balance the load is $F = -EI\, y''''$

$$\Rightarrow \rho \ddot{y} = -EI y''''$$

- ↳ at $x=0$, $y(0) = y'(0) = 0$ since this is a cantilever
- ↳ free end ∴ $B(L) = 0 \Rightarrow y''(L) = 0$

  $F(L) = 0 \Rightarrow y'''(L) = 0$ ← since $F = \frac{dB}{dx}$
- The equation can then be solved for $y(x,t) = y(x)e^{i\omega t}$

$$EIy'''' - \omega^2 \rho y = 0$$

$$\Rightarrow y = Ae^{ikx} + Be^{-ikx} + Ce^{kx} + De^{-kx}$$

↳ must be solved numerically for the modes.

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Fluid Dynamics

- In fluid, pressure increases with depth since more fluid must be supported. $P(z) = \rho g z$
  - ↳ hence a body with cross-section A experiences an upthrust $\rho g A \Delta z$.
  - ↳ this gives Archimedes' principle: the upthrust is equal and opposite to the weight of the fluid it displaces.
  - ↳ this buoyancy force acts through the centre of mass.

Ideal fluids

- An ideal fluid is incompressible and has no viscocity.
  - ↳ assume the mean free path λ of particles in the fluid is negligibly small
  - ↳ normal stresses decay so fast that the only possible stress is isotropic pressure $\sigma_1 = \sigma_2 = \sigma_3 = -p$
- The fluid is modelled as being composed of fluid elements.
- These elements have well-defined values of macroscropic properties like density, velocity, pressure.
- All fluids satisfy conservation of mass. The flux through an area element is $\rho \underline{v} \cdot d\underline{s}$, so the continuity equation is:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0}$$

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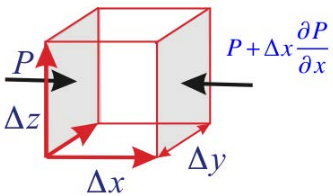
↳ for an incompressible fluid, $\rho = const \Rightarrow \boxed{\nabla \cdot \underline{v} = 0}$

· For a small fluid element, the variation in pressure causes acceleration.



$F_x = (\Delta y \Delta z)\left(-\Delta x \frac{\partial P}{\partial x}\right) = -V \frac{\partial P}{\partial x}$

↳ there is also the force of gravity

↳ so the equation of motion per unit volume is:

$$\boxed{\rho \frac{D\underline{v}}{Dt} = -\nabla P + \rho \underline{g}}$$ Euler's equation

· $\frac{D}{Dt}$ is the material derivative, necessary because velocity is treated as a function of space and time

↳ $d\underline{v} = dt \frac{\partial \underline{v}}{\partial t} + d\underline{x} \cdot \nabla \underline{v}$ but $d\underline{x} = \underline{v}\, dt$

$$\Rightarrow \boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla}$$

↳ i.e $\frac{D\underline{v}}{Dt}$ is the acceleration when moving in the same path as the fluid element

· Fluid flow can be visualised in 3 ways:

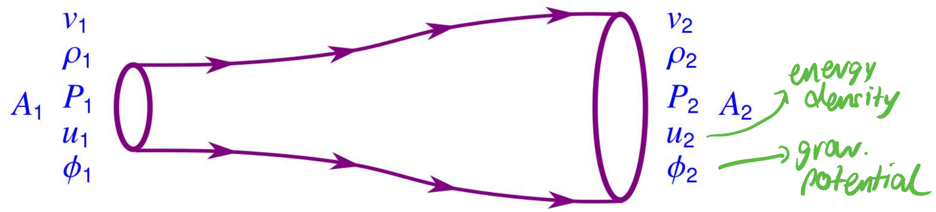
↳ pathlines track the movement of an element

↳ streamlines plot the velocity field at a given time

↳ streaklines connect all points that passed through a particular reference. e.g if a drop of dye were released at the reference

↳ all three coincide for steady flow

· For steady flow, Bernoulli's equation can be used to relate quantities along a streamline by conserving energy.



↳ energy flow in $= A_1 v_1 \left(\rho_1 \phi_1 + \frac{1}{2} \rho_1 v_1^2 + u_1 + P\right)$ = energy out

↳ by cons mass, $A_1 v_1 \rho_2 = A_2 v_2 \rho_2$, giving Bernoulli's equation

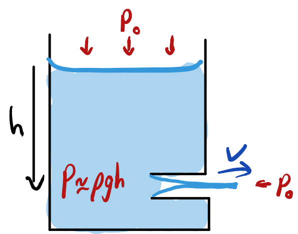
$$\boxed{\frac{u + P}{\rho} + \frac{1}{2} v^2 + \phi_g = \text{constant}}$$

↳ for incompressible flow, $u = 0$, $\rho = const$

$$\Rightarrow \boxed{P + \frac{1}{2} \rho v^2 + \rho \phi_g = const}$$

↳ curved streamlines require perpendicular pressure gradients to provide the centrifugal force.

e.g Borda's mouthpiece: what is the area of a jet of water from a deep hole?



· Momentum / time $= \rho v \cdot (v A_{jet})$

· By NII, $F = \frac{dp}{dt}$ ∴ $\rho g h A_{hole} = \frac{dp}{dt}$

$\Rightarrow \rho g h A_{hole} = \rho v^2 A_{jet}$

· But Bernoulli's equation gives $\rho g h = \frac{1}{2} \rho v^2$

$\Rightarrow A_{jet} = 0.5 A_{hole}$ → coefficient of efflux, usually 0.5→1.

Circulation

- In general, it is difficult to analyse fluids (even numerically) without assuming:
  - ↳ incompressible, $\nabla \cdot \underline{v} = 0$
  - ↳ irrotational, i.e no vorticity $\Rightarrow \underline{\omega} = \nabla \times \underline{v} = \underline{0}$.

    This is often reasonable in the bulk of the material.
- The circulation K around a loop Γ is defined as $K = \oint_\Gamma \underline{v} \cdot d\underline{l}$
  - ↳ related to vorticity by Stokes' theorem:

$$K = \oint_\Gamma \underline{v} \cdot d\underline{l} = \int \underline{\omega} \cdot d\underline{S}$$

  - ↳ Kelvin's circulation theorem states that the circulation around a loop moving with the fluid is constant. Proof:

$$\frac{DK}{DT} = \oint_\Gamma \left[\frac{D\underline{v}}{Dt} \cdot d\underline{l} + \underline{v} \cdot \frac{D(d\underline{l})}{Dt} \right]$$

use Euler's equation

rate of change of path must be $\nabla \underline{v} \cdot d\underline{l}$

$$= \oint_\Gamma \left[\nabla\left(-\frac{P}{\rho} - \phi_g \right) \cdot d\underline{l} + \underbrace{\underline{v} \cdot \nabla \underline{v} \cdot d\underline{l}}_{= \frac{1}{2} \nabla(v^2) \cdot d\underline{l}} \right]$$

$$\therefore \frac{DK}{Dt} = \oint_\Gamma \nabla\left(-\frac{P}{\rho} - \phi_g + \frac{1}{2} v^2 \right) \cdot d\underline{l}$$

  - ↳ since this quantity is single-valued, by the gradient theorem $\frac{DK}{Dt} = 0$.
  - ↳ i.e vortex lines are conserved and move with the fluid.
- We can then generalise Bernoulli's equation for an incompressible fluid. Using $\nabla\left(\frac{1}{2}\rho v^2\right) = \underline{v} \times (\nabla \times \underline{v}) + \underline{v} \cdot \nabla \underline{v}$ and $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \nabla$

$$\boxed{\nabla\left(P + \rho \phi_g + \frac{1}{2} \rho v^2 \right) = -\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \times (\nabla \times \underline{v})}$$

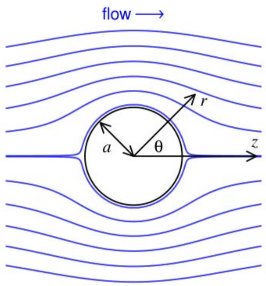
↳ for steady flow, $\frac{\partial \underline{v}}{\partial t} = 0$ so $\underline{v} \cdot \nabla\left(P + \rho\phi_g + \frac{1}{2}\rho v^2\right) = 0$.

Hence $P + \rho\phi_g + \frac{1}{2}\rho v^2 = \text{const}$ on a streamline (Bernoulli's equation)

↳ if steady and irrotational, $P + \rho\phi_g + \frac{1}{2}\rho v^2 = \text{const}$ everywhere.

Velocity potentials

- If $\nabla \times \underline{v} = \underline{0}$, $\underline{v} = \nabla \phi$ for some scalar velocity potential. If it is also incompressible, this potential satisfies Laplace's equation.
- Potential flow originates at a source/sink (analogous to charge).
  - ↳ if there is a flow rate Q: $\phi = -\frac{Q}{4\pi R}$ $\quad v = \frac{Q}{4\pi R^2} \hat{\underline{e}}_r$
  - ↳ we can apply the method of images to find ϕ, then $\underline{v} = \nabla \phi$, and pressure can be found with Bernoulli's equation.
  - ↳ hence a source and sink are repulsive.
- Analysing the flow past a sphere is analogous to a spherical conductor in an electric field.



  - ↳ $\phi = V_0 r\cos\theta$ far away and $v_r = 0$ at the boundary $r = a$
  - ↳ $\phi = V_0 r \cos\theta + \frac{B}{r^2}\cos\theta$ with $B = \frac{1}{2} V_0 a^3$
  - ↳ $V_\theta = -\frac{3}{2} V_0 \sin\theta$ at $r = a$
  - ↳ from Bernoulli: $P(\theta) + \frac{1}{2}\rho\left(\frac{3}{2} V_0 \sin\theta\right)^2 = P_0 + \frac{1}{2}\rho V_0^2$

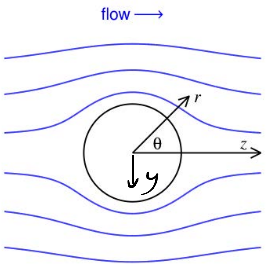
↳ pressure is symmetrical so there is no drag for this ideal fluid

↳ for sufficiently high velocities, $P(\theta) < 0$ at $\theta = \pm\frac{\pi}{2}$. This is unphysical – the fluid undergoes cavitation.

- For a cylinder, we have $\phi = V_0 \cos\theta \left(r + \frac{a^2}{r} \right)$
- But there is another flow: the vortex solution given by $\underset{\sim}{v} = \frac{\kappa}{2\pi r} \hat{\underset{\sim}{e}}_\theta$

↳ this is actually irrotational. $\oint_\Gamma \underset{\sim}{v} \cdot d\underset{\sim}{l} = 0$ if Γ does not contain the cylinder

↳ hence for a rotating cylinder with radius a and angular velocity $\kappa / 2\pi a^2$, $\phi = \frac{\kappa\theta}{2\pi}$ ← multivalued.



- For a rotating cylinder in a steady flow

$$\phi = V_0 \cos\theta \left(r + \frac{a^2}{r} \right) + \frac{\kappa\theta}{2\pi}$$

$$\Rightarrow V_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -2V_0 \sin\theta + \frac{\kappa}{2\pi a}$$

↳ from Bernoulli $P(\theta) + \frac{1}{2}\rho V_\theta^2 = P_0 + \frac{1}{2}\rho V_0^2$

$$\Rightarrow P(\theta) = P_0 + \frac{1}{2}\rho V_0^2 - \frac{1}{2}\rho \left[4V_0^2 \sin^2\theta + \frac{\kappa^2}{4\pi^2 a^2} - \frac{2V_0 \kappa \sin\theta}{\pi a} \right]$$

↳ because there is an asymmetric term in θ, there will be a net vertical force that can be found by integrating

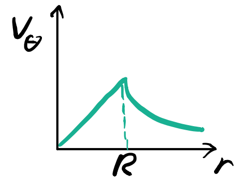
$$\frac{F_y}{L} = \int_0^{2\pi} \frac{\rho V_0 \kappa \sin\theta}{\pi a} \cdot a \sin\theta \, d\theta = \rho V_0 \kappa$$

↳ this is the Magnus force $\underset{\sim}{F} = \rho \underset{\sim}{V_0} \times \underset{\sim}{\kappa}$.

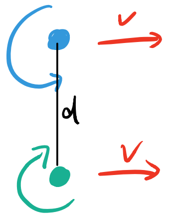
Vortices

- Vortices can appear in liquids without a solid rotating cylinder to cause them.
- The ideal irrotational vortex, with $\underset{\sim}{v} = \frac{\kappa}{2\pi r} \hat{\underset{\sim}{e}}_\phi$ has a singularity as $r \to 0$.
- The Rankine vortex model assumes a 'rigid body' rotating core of radius R, surrounded by a free vortex. This is similar to the B-field around a thick wire:

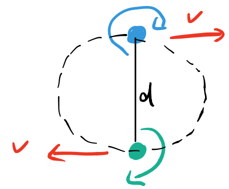
$$\boxed{V_\theta(r) = \begin{cases} \omega r, & r < R \\ \frac{\kappa}{2\pi r}, & r > R \end{cases} ; \quad \omega = \frac{\kappa}{2\pi R^2}}$$



- Thus two vortices of opposite sign will blow each other along at $v = \frac{\kappa}{2\pi d}$. Their separation is constant since the magnus force $\rho \underset{\sim}{v} \times \underset{\sim}{\kappa}$ is balanced by their attraction

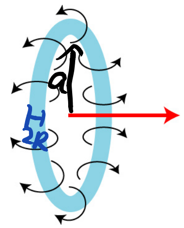


- Two vortices of the same sign will orbit around each other.



- We can construct a vortex ring (toroidal solenoid). Drifts at $\frac{\kappa}{4\pi a} \ln\left(\frac{a}{R}\right)$.

↳ near a flat plate, it interacts with its image and spreads out.



Real fluids

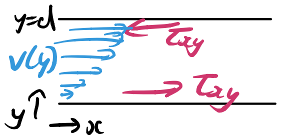
- Fluids cannot maintain a shear stress because molecules can move over each other.
  - ↳ a sudden shear ε_{xy} produces a stress that decays over a short timescale
  - ↳ to maintain a shear stress, it must be continuously sheared
  - ↳ for a Newtonian fluid, the strain rate is proportional to stress

$$\tau = \eta \frac{d\gamma}{dt} = 2\eta \frac{d\varepsilon_{xy}}{dt}$$ viscosity

- Viscosity depends on the spatial variation of velocity:

$$2\varepsilon_{xy} = \frac{\partial x}{\partial y} + \frac{\partial y}{\partial x} \Rightarrow 2\frac{d\varepsilon_{xy}}{dt} = \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}$$

  - ↳ viscosity is then defined as the shear flow between two flat plates at $y=0$, $y=d$



$$\boxed{\tau_{xy} = \frac{\text{Force}}{\text{Area}} = \eta \frac{\partial v_x}{\partial y}}$$

  - ↳ i.e. viscosity is the force per unit area, per unit velocity gradient.
  - ↳ η is related to the time it takes a shear stress to decay: $\eta = G t_r$.

- For a Newtonian fluid, $\tau_{ij} = \eta\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$
  - ↳ force/volume comes from varying shear stresses,

$$\sum_j \frac{\partial \tau_{ij}}{\partial x_j} = \eta \sum_j \left(\frac{\partial^2 v_i}{\partial^2 x_j} + \frac{\partial^2 v_j}{\partial x_i \partial x_j}\right)$$

  - ↳ in vector form, the new equation of motion:

$$\boxed{\rho \frac{D\underline{v}}{Dt} = -\nabla P + \rho \underline{g} + \eta\left(\nabla^2 \underline{v} + \nabla(\nabla \cdot \underline{v})\right)}$$

  - ↳ there is actually a constant in front of $\nabla(\nabla \cdot \underline{v})$ related to the bulk modulus since there is resistance to volume change.
  - ↳ for compressible fluids, this simplifies to:

$$\rho \frac{D\underline{v}}{Dt} = -\nabla P + \rho \underline{g} + \eta \nabla^2 \underline{v}$$

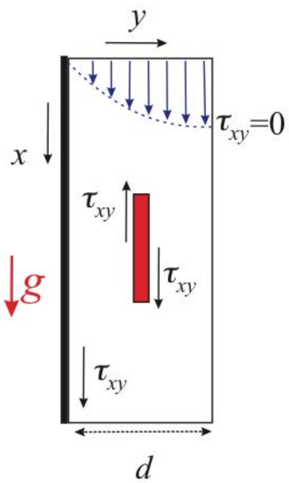
- Microscopically, viscosity depends on the collisional mean free path. Consider a space-varying quantity Q
  - ↳ as a particle moves over distance λ_c at velocity v_T (the thermal velocity), it exchanges ΔQ with surroundings
  - ↳ this random walk leads to a diffusion equation

$$\frac{DQ}{Dt} = \frac{1}{3}\lambda_c v_T \nabla^2 Q$$ due to 3D motion

  - ↳ for viscosity, $Q = \rho \underline{v}$.
  - ↳ we define the kinematic viscosity $\nu \equiv \frac{\eta}{\rho} = \frac{1}{3}\lambda_c v_T$

Poiseuille Flow

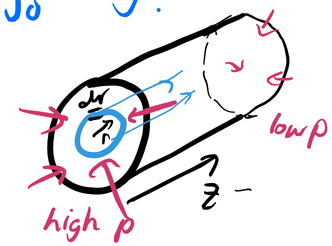
- Consider shear flow for a draining plate.
  - ↳ for steady flow, $\frac{D\underline{v}}{Dt}=0$
  - ↳ $p=p_0$ everywhere $\Rightarrow \nabla p = 0$
  - ↳ $\underline{v}=v_x\hat{\underline{x}}$ and only varies with y
  - ↳ no slip at $y=0 \Rightarrow v_x = 0$
  - ↳ no shear at $y=d \Rightarrow \tau_{xy}=\eta\frac{dv_x}{dy}$
  - ↳ hence the equation of motion for an incompressible fluid gives



$$\eta\frac{d^2v_x}{dy^2}=-\rho g \Rightarrow v_x=\frac{\rho g}{\eta}\left(yd-\frac{1}{2}y^2\right)$$

- ↳ this gives Poiseuille flow (parabolic)
- ↳ total flow rate per unit area: $Q=\int_0^d v_x\,dy$.

- Consider flow in a circular pipe with a pressure gradient.
  - ↳ for a annular cylindrical element between $r\rightarrow r+dr$ with length l



$$F_z=-\frac{dP}{dz}l\cdot 2\pi r\,dr$$

- ↳ must be balanced by viscous forces for steady flow

$$\tau_{xy}=\eta\frac{dv_z}{dr}\Rightarrow F_v=2\pi r l\eta\frac{dv_z}{dr}$$

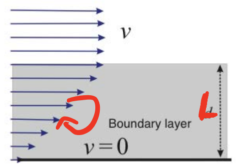
- ↳ the net viscous force is $\frac{dF_v}{dr}dr$

$$\therefore\ \frac{d}{dr}\left(2\pi r l\,\eta\frac{dv_z}{dr}\right)=\frac{dP}{dz}2\pi r l$$

$$\Rightarrow \boxed{v_z=\frac{1}{4\eta}\left|\frac{dP}{dz}\right|(a^2-r^2)}$$

Boundary layers and the Reynolds number

- The bulk of a liquid may have steady flow, but to match the no-slip B.C, there must be a boundary layer in which there is a velocity gradient and vorticity.



- The Reynolds number is the ratio of inertial stress to viscous stress (due to shear forces)
  - ↳ the inertial stress is $F/A=\frac{1}{A}\frac{dp}{dt}=\rho v^2$
  - ↳ the viscous stress for linear velocity change is

$$\tau_{xy}=\eta\frac{dv_x}{dy}=\eta v/L$$

  - ↳ the Reynolds number is then: $\boxed{N_R=\frac{\rho v L}{\eta}}$

- If N_R is large (i.e high inertial stress), random transverse motions (eddy flows) cause turbulence, increasing the effective viscosity: $(\eta/\rho)_{effective}\approx\lambda_c v_T+L_{eddy}v_{eddy}$

- Fluid flow around a sphere is complicated but can be analysed with dimensional analysis (for simple fluids)
- The drag force must be a function of $\{\rho,\eta,v_0,d\}$, i.e some force $\times$ dimensionless function

Robert Andrew Martin

$\Rightarrow \boxed{F = \rho v_0^2 d^2 \times C_D(N_R)}$

↳ C_D is the drag coefficient, a function of the Reynolds number.

↳ for low N_R, viscosity dominates so $F \propto \eta d v_0$,
i.e $C_D \propto 1/N_R$

↳ for high N_R, inertial effects dominate so $F \propto \rho d^2 v_0^2$,
i.e C_D is constant