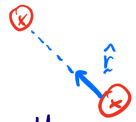
Robert Andrew Martin

Electromagnetism

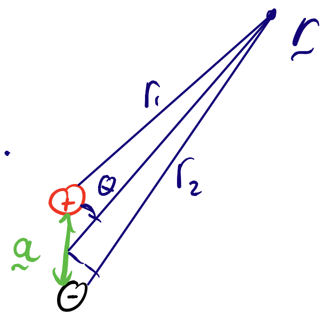
Electrostatic Fields

- Coulomb showed that $\underset{\sim}{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\underset{\sim}{r}}$



- The potential difference between two points is the work per unit charge to move a small test charge from A to B. $V_{BA} = -\int_A^B \underset{\sim}{E}(\underset{\sim}{r})\cdot d\underset{\sim}{r}$
  - ↳ alternatively, $\underset{\sim}{E}(\underset{\sim}{r}) = -\nabla V(\underset{\sim}{r})$
  - ↳ V only defined to within a constant offset – this constant is the gauge of the field.
  - ↳ by summing the p.d.s across a small loop, we can derive $\nabla \times \underset{\sim}{E} = 0$. ← essentially a consequence of Stokes' theorem
  - ↳ potentials add by linear superposition
- We can analyse how potentials vary in space using Poisson's equation $\nabla^2 V(\underset{\sim}{r}) = -\frac{\rho(\underset{\sim}{r})}{\varepsilon_0}$
  - ↳ we can solve for V if we know $\rho(\underset{\sim}{r})$ and the B.C.s
  - ↳ when there is no charge, Poisson → Laplace's equation.
  - ↳ we can either specify the quantity (Dirichlet), normal derivative of the quantity (Neumann) or both (Cauchy) as boundary conditions
- B.C.s guarantee uniqueness to within an additive const.
  - ↳ can be shown by assuming there exist two solutions and examining $\Phi(\underset{\sim}{r}) = V(\underset{\sim}{r}) - U(\underset{\sim}{r})$
  - ↳ then use identity $\nabla\cdot(\phi\nabla\phi) = |\nabla\phi|^2 + \phi\nabla^2\phi$ and integrate both sides over the total volume
  - ↳ $\phi = 0$ on boundary and $\nabla\phi = 0$ everywhere $\Rightarrow U = V$
- Poisson's equation can be solved with the Green's function, the solution of $\nabla^2 G(\underset{\sim}{r}, \underset{\sim}{r}') = -\delta(\underset{\sim}{r} - \underset{\sim}{r}')$ which satisfies homogeneous B.Cs, i.e. $aG + bG' = 0$ at each point on the boundary
  - ↳ once G is known, we can find the potential via $V(\underset{\sim}{r}) = \frac{1}{\varepsilon_0}\int_{V'} G(\underset{\sim}{r}, \underset{\sim}{r}')\rho(\underset{\sim}{r}')\,dV'$

Dipoles



- A monopole is a single point charge q.
- At large distances away:

$$r_1 \approx r - \tfrac{a}{2}\cos\theta$$
$$r_2 \approx r + \tfrac{a}{2}\cos\theta$$

- Then, the dipole potential is $V(\underset{\sim}{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$
  - ↳ expand in a/r and let $\underset{\sim}{p} \equiv q\underset{\sim}{a}$ be the electric dipole moment to get $\boxed{V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$

· The gradient in spherical coordinates is:

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}$$

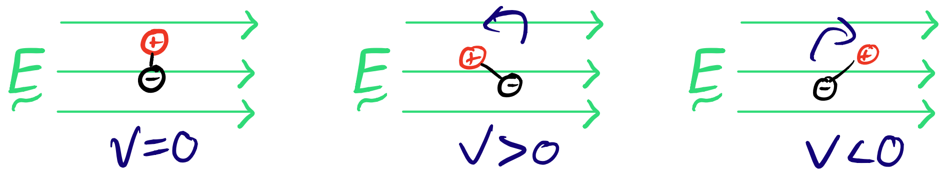
· Hence the electric field of a dipole is:

$$\underline{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

· If a dipole is placed in a uniform field, it experiences a couple: $\boxed{\underline{G} = \underline{p}\times\underline{E}}$ $\quad (\underline{G} = \underline{r}\times\underline{F} = q\underline{r}\times\underline{E}/q)$

↳ $|\underline{G}| = p|\underline{E}|\sin\theta$ ⇒ couple is zero when dipole aligned to $\underline{E}$

↳ a dipole thus has potential energy since work is done to rotate it in a field $(dW = |\underline{G}(\theta)|\,d\theta)$

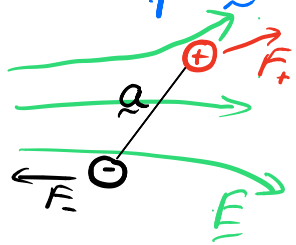


↳ with this (arbitrary) convention: $U = -\underline{p}\cdot\underline{E}$

· When the field is non-uniform, we can Taylor-expand the field:

$$E_x^+ \simeq E_x^- + \underline{a}\cdot\nabla E_x$$

and $F_x = q(E_x^+ - E_x^-)$



↳ repeating this for y and z, we can show that:

$\underline{F} = (\underline{p}\cdot\nabla)\underline{E}$ ⇐ NB: grad of a vector field

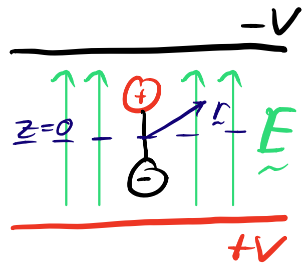
↳ if $\underline{p}$ is constant, we can say:

$\underline{F}(\underline{r}) = \nabla(\underline{p}\cdot\underline{E}(\underline{r})) = -\nabla U(\underline{r})$ ← rotational PE

↳ in reality, the dipole will move and $\underline{F}$ must be recalculated

· Consider a dipole within a uniform field. The potential at some point is:

$$V(\underline{r}) = -E_0 r\cos\theta + \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$



↳ there is thus a spherical surface for which $V=0$.

↳ we could replace this with a spherical conductor and still satisfy the B.Cs – by uniqueness, this must be the solution.

· Thus a uniform conductor in a field is equivalent to a dipole ⇒ induced dipole, with moment:

$$p = \underbrace{4\pi\varepsilon_0 a^3}_{\alpha} E_0$$

α is the polarisability

↳ then we have $\underline{p} = \alpha\underline{E}_0$

↳ the induced charge is not 'because of' the dipole

↳ in general, α will be a tensor because $\underline{p}$ and $\underline{E}_0$ need not align.

· We can now calculate $V(\underline{r})$ and $\underline{E}(\underline{r})$. The surface charge density of the sphere is given by: $E_r = \frac{\sigma}{\varepsilon_0}$

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- In principle, we can analyse more complex charge distributions using *multipole expansions*.
  - ↳ e.g a *quadrupole* field drops off as $1/r^3$

  - ↳ multipole potentials form a complete set of functions

The divergence of E fields

- *Electric flux* is a mathematical concept related to the density of field lines through a patch: $\int_S \underline{E} \cdot d\underline{s}$
- From the divergence theorem: $\oint_S \underline{E} \cdot d\underline{s} = \int_V \nabla \cdot \underline{E} \, dV$
- Div. theorem can be combined with charge conservation
  - ↳ consider some volume with a current density $\underline{J}$ at some point on the surface

  - ↳ by cons. charge:

$$I = \oint_S \underline{J} \cdot d\underline{s} = -\frac{\partial}{\partial t} \int_V \rho \, dt = \int_V \nabla \cdot \underline{J} \, dV$$

div. theorem

$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \underline{J} = 0}$$

- It can also be used to derive *Gauss' law* by considering the flux from a point charge. Define the *electric displacement* of free space $\underline{D}(\underline{r}) = \varepsilon_0 \underline{E}(\underline{r})$

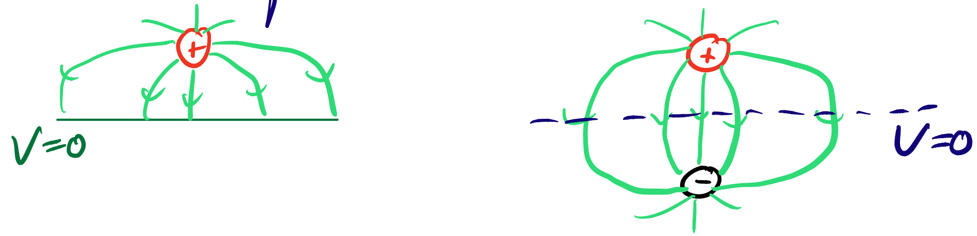
$$\therefore \oint_S \underline{D}(\underline{r}) \cdot d\underline{s} = Q_{enc} \iff \nabla \cdot \underline{D}(\underline{r}) = \rho(\underline{r})$$

actually this is only free charge

The method of images

- Generalises the approach used to analyse spherical conductor in $\underline{E}$ field.
- By uniqueness, if we can construct some charge dist. that fits the B.Cs, that solution is <u>the</u> solution.
- The method of images can be used to calculate potentials and fields in the presence of a conductor.

e.g



  - ↳ we can introduce image charges such that the potential is equivalent to if there were a conductor.
  - ↳ these images must not be in the same region where you want to calculate potential.
- The surface charge density on the conductor can be calculated using the B.C $\sigma = \varepsilon_0 E_\perp$
- The energy of an image arrangement can be found using $W = \frac{1}{2}\iiint \rho V \, d\tau = \frac{1}{2}\sum q_i V_i$
  - ↳ be careful when integrating because image charges move too.
- The image for a line charge parallel to a cylindrical conductor is a line charge inside the cylinder.

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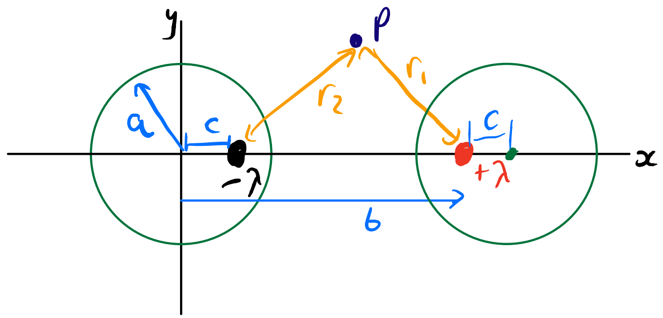
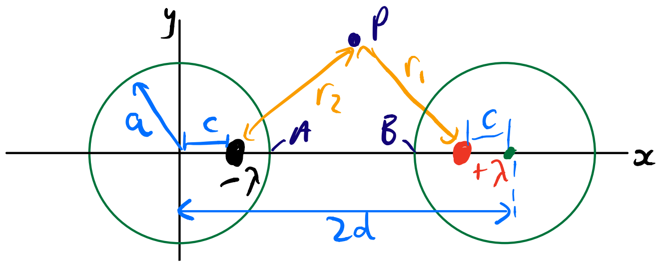


- $V(\underline{r}) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{r_2}{r_1}\right) \Rightarrow \frac{r_2^2}{r_1^2} = K = \text{const}$ for equipotentials.
  - ↳ expanding out shows that $c = a^2/b$

Capacitance

- The capacitance of a two-surface metallic structure is the amount of net positive charge on the high-potential surface divided by the p.d. $C = Q/V$
- Consider the system of two parallel cylindrical conductors, with separation $2d$.



↳ $c = a^2/b = a^2/2d-c \quad \therefore \quad c = d - \sqrt{d^2-a^2}$

↳ we can choose easy points A, B to evaluate V

↳ the potential is then $V = \frac{\lambda}{\pi\varepsilon_0}\ln\left(\frac{a}{d-\sqrt{d^2-a^2}}\right)$

↳ in the $d \gg a$ limit: $C \approx \frac{\pi\varepsilon_0}{\ln(2d/a)}$

Electrostatic energy

- By building up a set of charges, it can be shown that:

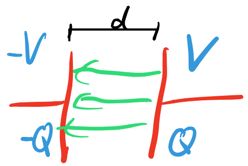
$$U = \frac{1}{2}\sum_{i=1}^{N} q_i V_i \leftarrow V_i \text{ is the potential at position } i \text{ WITHOUT ith charge}$$

- In the continuous case:

$$U = \frac{1}{2}\iiint \rho(\underline{r})\,V(\underline{r})\,d\tau$$

$V(\underline{r})$ must exclude the charge element at $\underline{r}$.

- Consider a parallel-plate capacitor



↳ let $Q = \int dQ$ and by excluding the charge element we have $V = \frac{Q-dQ}{C}$

↳ the energy in the capacitor is:

$$U = \frac{1}{2}\int V\,dQ = \frac{1}{2}\int \frac{Q-dQ}{C}\,dQ \quad \rightarrow O((dq)^2) \to 0$$

$$\therefore U = \frac{1}{2}QV = \frac{1}{2}CV^2$$

- We can also derive this by considering the field:

$$U = \frac{1}{2}QV = \frac{1}{2}(A\varepsilon_0|\underline{E}|)(|\underline{E}|d)$$

$$\therefore \boxed{U_E(\underline{r}) = \frac{1}{2}\varepsilon_0|\underline{E}(\underline{r})|^2 = \frac{1}{2}\underline{D}(\underline{r})\cdot\underline{E}(\underline{r})}$$

energy density

↳ then $U = \iiint U_E(\underline{r})\, d\tau$

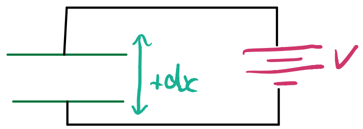
↳ we could also have found this using

$$U = \frac{1}{2}\iiint \rho(\underline{r})\, V(\underline{r})\, d\tau = \frac{1}{2}\varepsilon_0 \iiint (\nabla \cdot \underline{E})\, V(\underline{r})\, d\tau$$

then applying an identity.

Forces on charge distributions

- The method of virtual work calculates the force by considering how a small perturbation dx changes the energy of the system.
- Consider a capacitor held at constant potential, with a perturbation dx increasing the separation



↳ the electric field (and thus energy) decreases with separation.

$$U_s = \frac{1}{2}\varepsilon_0 \frac{V^2}{x} A \quad \therefore\ dW = \frac{\partial U_s}{\partial x} dx = -\frac{1}{2}\varepsilon_0 \frac{V^2}{x^2} A\, dx$$

↳ but the decrease in charge on the plates causes power dissipation in the battery

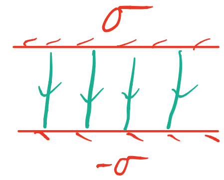
$$dW = -V\, dQ\,, \quad Q = \frac{V}{x}\varepsilon_0 A \quad \therefore\ dW = \varepsilon_0 \frac{V^2}{x^2} A\, dx$$

↳ considering both of these energy changes:

$$F\, dx = -\frac{1}{2}\varepsilon_0 \frac{V^2}{x^2} A\, dx + \varepsilon_0 \frac{V^2}{x^2} A\, dx$$

$$\therefore\ F = \frac{1}{2}\varepsilon_0 V^2 \frac{A}{x^2}$$

- The electric field between two charged conductors is $|\underline{E}| = \frac{\sigma}{\varepsilon_0}$



- However, to find the force (without virtual work), we must exclude the current plate.

$$\therefore\ F = Q|\underline{E}| = \sigma A \cdot \frac{\sigma}{2\varepsilon_0} = Q^2/2\varepsilon_0 A$$

↳ this is an attractive force obviously

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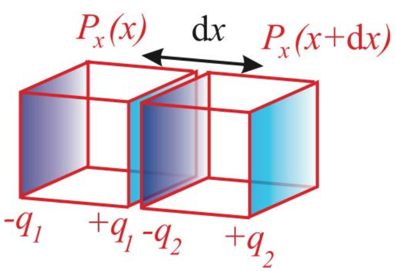
Dielectrics

- When an insulator (a.k.a dielectric) is placed in an electric field, dipole moments are induced. These are *bound charges*, as opposed to the *free charges* in conductors.
  - ↳ this charge only appears on surfaces because internally the bound charges cancel.
  - ↳ this is quantified by the *polarisation*, which is the dipole moment per unit volume

number density of atoms

$$\underset{\sim}{P} = N \underset{\sim}{p} \implies |P| = \frac{Q}{A}$$

- With a dielectric in a capacitor at fixed potential, the free charge on the plates must increase to compensate for the bound charge at the surface.
  - ↳ the charge increases by a factor of ε_r, which is the *relative permittivity* of the dielectric.
  - ↳ $\therefore C = Q/V = \varepsilon_r \varepsilon_0 A / d$. Q is only the free charge!
- If the electric field is non-uniform, there will be some polarisation charge density within the dielectric



$$q = q_1 - q_2 = [P_x(x) - P_x(x+dx)]\,dy\,dz$$
$$= -\frac{\partial P_x}{\partial x}\,dx\,dy\,dz$$

$$\therefore \boxed{\rho_b = -\nabla \cdot \underset{\sim}{P}(\underset{\sim}{r})} \quad \leftarrow \rho_b \text{ is the bound/polarisation charge.}$$

- In general, some volume of space may contain both free and bound charges. Applying Gauss' law:

$$\int_S \underset{\sim}{E} \cdot d\underset{\sim}{S} = \frac{1}{\varepsilon_0}\int_V \rho_{total}\, d\tau = \frac{1}{\varepsilon_0}\int_V \rho_f + \rho_b \, d\tau$$

$$\implies \int_V \nabla \cdot \underset{\sim}{E}\, d\tau = \frac{1}{\varepsilon_0}\int_V \rho_f - \nabla \cdot \underset{\sim}{P}\, d\tau$$

$$\implies \nabla \cdot [\varepsilon_0 \underset{\sim}{E} + \underset{\sim}{P}] = \rho_f$$

  - ↳ we define the *electric displacement field* $\boxed{\underset{\sim}{D} = \varepsilon_0 \underset{\sim}{E} + \underset{\sim}{P}}$

$$\therefore \boxed{\nabla \cdot \underset{\sim}{D} = \rho_f} \quad \leftarrow \text{free charge is the source of } \underset{\sim}{D}$$

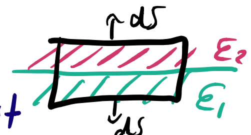
  - ↳ in a linear dielectric (e.g if $\underset{\sim}{E}$ is small), polarisation is proportional to the field where the constant is the *susceptibility*, $\underset{\sim}{P} = \varepsilon_0 \chi \underset{\sim}{E}$. Using $\varepsilon_r = 1 + \chi$:

$$D(\underset{\sim}{r}) = \varepsilon_r \varepsilon_0 E(\underset{\sim}{r})$$

- The original Maxwell equation $\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0}$ is still valid, but formulating in terms of $\underset{\sim}{D}$ is easier because it absorbs the effects of the dielectric into a constant.
- For electrostatics problems with fixed potentials, the electric field will be the same because of the uniqueness theorem
  ① $\underset{\sim}{E}$ from V (Poisson's eq, $E = -\nabla V$) ② $\underset{\sim}{D}$ from $\underset{\sim}{E}$ ($\underset{\sim}{D} = \varepsilon_r \varepsilon_0 \underset{\sim}{E}$) ③ σ from $\underset{\sim}{D}$ (Gauss' law)
- If charge is fixed (i.e ρ_f known) on all surfaces
  ① $\underset{\sim}{D}$ from ρ_f (Gauss' law) ② $\underset{\sim}{E}$ from $\underset{\sim}{D}$ ($\underset{\sim}{E} = \underset{\sim}{D}/\varepsilon_r \varepsilon_0$) ③ V from $\underset{\sim}{E}$ ($V = -\int \underset{\sim}{E} \cdot d\underset{\sim}{l}$)

Boundary conditions

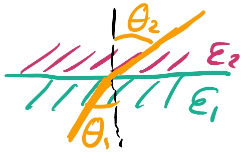
- To understand the B.C.s, we can construct a Gaussian pillbox. Since $\rho_f = 0$, we must have $D_{2\perp} = D_{1\perp}$. $D = \varepsilon_r \varepsilon_0 E$, so $E_\perp$ is discontinuous.



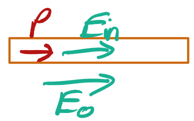
- Constructing a loop and using $\oint \underline{E} \cdot d\underline{l} = 0$, we must have $E_{1\parallel} = E_{2\parallel}$ $\therefore$ $D_\parallel$ discontinuous.

- Hence, though D and E are everywhere parallel to each other, the direction of the field lines changes at a boundary:

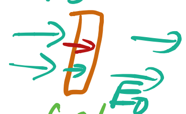
$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$



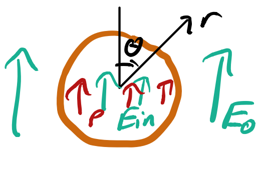
- For simple geometries, B.C.s can be applied directly to find $\underline{P}$ (normally the quantity of interest).
- Long thin rod parallel to uniform field:
  ↳ $E_\parallel$ cont. $\therefore$ $E_{in} = E_0$ $\therefore$ $P = \varepsilon_0 \chi E$



- Thin slab perpendicular to uniform field:
  ↳ $D_\perp$ cont $\therefore$ $E_{in} = \frac{1}{\varepsilon_r} E_0$
  $\Rightarrow P = \varepsilon_0 E_0 \frac{\chi}{1+\chi}$ ← this form of relationship is very common



- Dielectric sphere in uniform field:
  ↳ guess that internal field is uniform and externally there is some dipole field due to surface polarisation charge: $V_{in} = -E_{in} r\cos\theta$
  $V_{out} = -E_0 r\cos\theta + \frac{A\cos\theta}{r^2}$



↳ $V_{in} = V_{out}$ at boundary (equivalent to $E_\parallel$ cont)
↳ $D_\perp$ cont $\therefore$ $-\varepsilon_0\varepsilon_r \frac{\partial V_{in}}{\partial r}\Big|_{r=a} = -\varepsilon_0 \frac{\partial V_{out}}{\partial r}\Big|_{r=a}$
↳ applying these B.Cs gives $E_{in} = \frac{3}{\varepsilon_r+2} E_0$
$\Rightarrow P = \frac{\chi}{1+\chi/3} \varepsilon_0 E_0$
↳ by uniqueness, this is the solution.

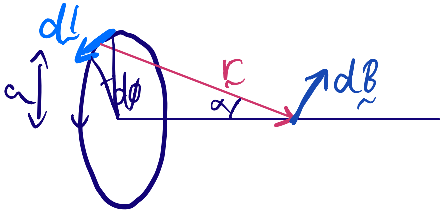
Magnetostatics

- A current element is an infinitesimal wire filament $d\mathbf{l}$ carrying current I.
- The magnetic field $\mathbf{B}$ is defined as

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B}$$

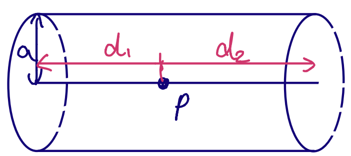
- The B-field produced by a current element is given by the Biot-Savart law: $\boxed{d\mathbf{B} = \frac{\mu_0 I}{4\pi r^2}\, d\mathbf{l} \times \hat{\mathbf{r}}}$
  - ↳ μ_0 is the permeability of free space
  - ↳ B-field is inverse square and field lines circulate
- The force between two current elements can thus be evaluated: $d\mathbf{F} = \frac{I_1 I_2 \mu_0}{4\pi r^2}\, d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \hat{\mathbf{r}})$
  - ↳ greatest when elements are aligned
  - ↳ attractive when currents flow in the same direction
  - ↳ can be used to define the ampere.

e.g the on-axis field of a current loop



$$B_x = \frac{\mu_0 I}{4\pi r^2} \sin\alpha \oint dl$$
$$= \frac{\mu_0 I a^2}{2 r^3}$$

e.g the on-axis field of a solenoid with n loops / unit length



$$dB = \frac{\mu_0 I a^2}{2 r^3} \cdot n\,dx$$
$$\Rightarrow B = \frac{\mu_0 n I}{2}\left[\frac{x}{\sqrt{a^2+x^2}}\right]_{-d_1}^{d_2}$$

↳ thus for a long solenoid, $d_1, d_2 \to \infty$ and hence $\mathbf{B} = \mu_0 n I$

- Magnetic field lines form closed loops and wrap around electrical currents. Magnetic monopoles do not exist.
  - ↳ thus the net magnetic flux (Φ) through a closed surface is zero.

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \iff \nabla \cdot \mathbf{B} = 0}$$

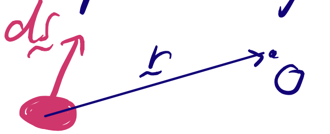
Magnetic dipoles

- A magnetic dipole is modelled as a small current loop with vector area $d\mathbf{S}$
  - ↳ the dipole moment is $d\mathbf{m} = I\, d\mathbf{S}$
  - ↳ the torque on this loop in a field is $d\mathbf{G} = d\mathbf{m} \times \mathbf{B}$
- For an arbitrarily-shaped loop in a field, the net torque is $\mathbf{G} = \mathbf{m} \times \mathbf{B}$, with $\mathbf{m} = I \int_S d\mathbf{S}$
  - ↳ S is any surface bounded by the loop.

- The potential energy of a magnetic dipole is given by $U = -\underline{m}\cdot\underline{B}$ ← identical to electric dipole
  ↳ a macroscopic current loop can be constructed from many magnetic dipoles:
  $$U = -\int \underline{B}\cdot d\underline{m} = -\int_S I\,d\underline{S}\cdot\underline{B} = -I\phi$$
- In a non-uniform field, the force on a dipole (assuming dipole moments are constant in space) is given by $\underline{F}(\underline{r}) = \nabla(\underline{m}\cdot\underline{B}(\underline{r}))$

Magnetic scalar potential

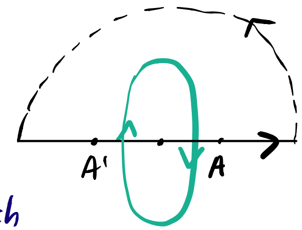
- Analogous to the electric potential, it is useful to define
  $$H(\underline{r}) = -\nabla\phi_m(\underline{r})$$
  ↳ H is the magnetic field strength, from which we can get the flux density $\underline{B} = \mu\underline{H}$
- We can calculate ϕ_m for a loop with the concept of solid angle
- The solid angle subtended by some surface element is given by:
  $$d\Omega = \frac{|d\underline{S}|}{r^2}\cos\theta$$



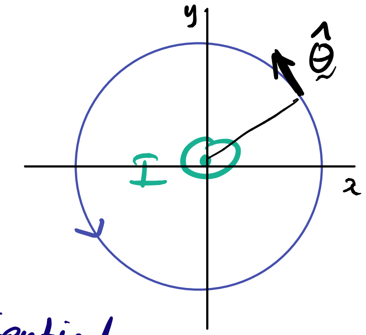
NB: inwards surface normal and $\underline{r}$ points towards origin

- The magnetic/electric dipole fields have the same form, so for a magnetic dipole: $\phi_m(r,\theta) = \dfrac{|d\underline{m}|\cos\theta}{4\pi r^2}$

- Using $|d\underline{m}| = I|d\underline{S}|$, and breaking down an arbitrary loop into infinitesimal ones: $\boxed{\phi_m = \dfrac{I\Omega}{4\pi}}$
  Ω is the solid angle subtended →
- If we consider traversing some arbitrary closed path through a loop, we notice a discontinuity in ϕ_m at the centre of the loop.
  ↳ $\Omega = 2\pi$ at A, $\Omega = -2\pi$ at A'
  $$\therefore \int_A^{A'} d\phi_m = -I$$



  ↳ but $\int_A^{A'} d\phi_m = \int_A^{A'} \nabla\phi_m\cdot d\underline{l}$, which in the limit becomes $-\frac{1}{\mu_0}\oint\underline{B}\cdot d\underline{l}$
  ↳ thus we have derived Ampere's law:
  $$\boxed{\oint\underline{B}\cdot d\underline{l} = \mu_0 I}$$
- We can find I as $I = \int\underline{J}\cdot d\underline{S}$. Using Stokes' theorem: $\boxed{\nabla\times\underline{H} = \underline{J}}$
  ↳ by Ohm's law: $\boxed{\underline{J} = \sigma\underline{E}}$



- For a long wire, the B field is: $\underline{B} = \frac{\mu_0 I}{2\pi r}\hat{\underline{\theta}}$
- In cylindrical polars, $(\nabla)_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}$
  ∴ for a wire, $\phi_m = -I\theta/2\pi$
  ↳ i.e. increasing θ decreases potential

- However, because $\nabla\times\underset{\sim}{B}\neq\underset{\sim}{0}$, the magnetic potential must be multivalued, e.g. at $\theta=0, 2\pi, 4\pi, \ldots$

Magnetic vector potential

- We know that $\nabla\cdot\underset{\sim}{B}=0$, so it is reasonable to assume that we can write $\underset{\sim}{B}(\underset{\sim}{r})=\nabla\times A(\underset{\sim}{r})$
- However, it can be seen that $\underset{\sim}{A}$ is undefined to within a radial vector field $\underset{\sim}{K}(\underset{\sim}{r})$, where

  $\underset{\sim}{K}(\underset{\sim}{r})=(K_x(x), K_y(y), K_z(z))$ ← e.g. $\frac{\partial k_x}{\partial y}=0$

  ↳ $\nabla\cdot\underset{\sim}{A}$ can be arbitrarily set without affecting the curl – this is known as choosing the gauge.

  ↳ a common choice is $\nabla\cdot\underset{\sim}{A}(\underset{\sim}{r})=0$ everywhere

- Then $\nabla\times\underset{\sim}{B}=\mu_0\underset{\sim}{J}\Rightarrow\boxed{\nabla^2\underset{\sim}{A}=-\mu_0\underset{\sim}{J}}$

  ↳ this is analogous to Poisson's equation, and we can conclude that $\underset{\sim}{A}(\underset{\sim}{r})=\frac{\mu_0}{4\pi}\int\frac{\underset{\sim}{J}(\underset{\sim}{r}')}{|\underset{\sim}{r}-\underset{\sim}{r}'|}d^3\underset{\sim}{r}'$

  ↳ we thus have a general way to compute the B field, though in practice Biot-Savart or Ampère's laws are more useful.

Magnetic fields in matter

- The magnetic dipole of an atom fundamentally arises from the orbital motion of electrons and their spin.
- However, we model magnetisation in terms of current loops.
- Magnetisation ($\underset{\sim}{M}$) is the magnetic dipole moment per unit volume – analogous to $\underset{\sim}{P}$ in electrostatics

  ↳ the total magnetic dipole moment of an object can be found by integration: $\underset{\sim}{m}_t=\int_V\underset{\sim}{M}\,d\tau$

  ↳ $\underset{\sim}{M}$ can be associated with the fictitious magnetisation current: $\underset{\sim}{J}_m=\nabla\times\underset{\sim}{M}$ ← analogous to $\rho_b=-\nabla\cdot\underset{\sim}{P}$

  ↳ If an object has uniform magnetisation, $\underset{\sim}{J}_m$ must reside on the surface (inner loops cancel out). The surface current density $\underset{\sim}{J}_s=\underset{\sim}{M}\times\hat{\underset{\sim}{n}}$

  ↳ analogous to $\sigma_p=\underset{\sim}{P}\cdot\hat{\underset{\sim}{n}}$

- In a magnetic material, we can consider that the resulting flux density is a result of a 'free space' field strength and the magnetisation: $\underset{\sim}{B}=\mu_0(\underset{\sim}{H}+\underset{\sim}{M})$

  ↳ $\underset{\sim}{H}$ is the magnetic field strength, such that

  $$\boxed{\nabla\times\underset{\sim}{H}=\underset{\sim}{J}_{free}\quad\text{and}\quad\oint\underset{\sim}{H}(\underset{\sim}{r})\cdot d\underset{\sim}{l}=I}$$

  ↳ these relations do not depend on the material, though to calculate forces we ultimately need $\underset{\sim}{B}$.

- For many materials (and small field strengths), $\underset{\sim}{M}$ scales linearly with $\underset{\sim}{H}$, i.e $\underset{\sim}{M} = \chi_m \underset{\sim}{H}$ where χ_m is the magnetic susceptibility of the material.

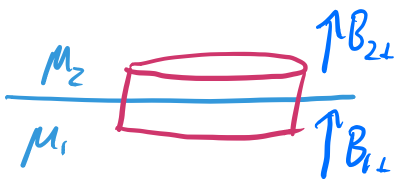
$$\therefore \quad \underset{\sim}{B} = \mu_0(1+\chi_m)\underset{\sim}{H} \equiv \mu_0\mu_r \underset{\sim}{H}$$

↳ diamagnetic $\chi_m < 0$ (small negative)
↳ paramagnetic $\chi_m > 0$ (small positive)
↳ ferromagnetic $\chi_m \gg 0$

- Some materials are permanently magnetied, with a constant magnetic dipole independent of $\underset{\sim}{H}$.

Inhomogeneous magnetic materials

- Consider a pillbox on a boundary between two magnetic materials.



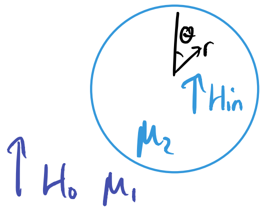
$$\oint_S \underset{\sim}{B}\cdot d\underset{\sim}{S} = 0 \quad \Rightarrow \quad \underset{\sim}{B}_{1\perp} = \underset{\sim}{B}_{2\perp}$$

↳ normal component of $\underset{\sim}{B}$ is continuous across the boundary (and thus $\underset{\sim}{H}_\perp$ is discontinuous)
↳ by setting up a loop, we can show that

$$\oint \underset{\sim}{H}\cdot d\underset{\sim}{l} = 0 \quad \Rightarrow \quad \underset{\sim}{H}_{1\parallel} = \underset{\sim}{H}_{2\parallel}$$

- This is very similar to the electrostatics case, with $\underset{\sim}{B} \sim \underset{\sim}{D}$, and $\underset{\sim}{H} \sim \underset{\sim}{E}$
↳ H, E are field strengths. B, D are flux densities.

- The B.Cs can be used to find the fields in magnetisable objects: e.g a sphere in a uniform field



Let $\underset{\sim}{H} = -\nabla\phi_m$. Assume uniform field

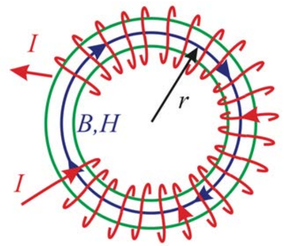
$$\therefore \quad \phi_m(r) = \begin{cases} -H_{in}\, r\cos\theta & r<a \\ -H_0\, r\cos\theta + \frac{A\cos\theta}{r^2}, & r>a \end{cases}$$

$H_\parallel$ cont and $B_r = -\mu\mu_0 \frac{\partial\phi_m}{\partial r}$ cont

$$\Rightarrow H_{in} = \frac{3}{\mu+2} H_0$$

Electromagnets

- Consider a toroidal solenoid (high permeability) with N turns of wire.
↳ we can analyse the system with an Amperian loop:



$$\oint \underset{\sim}{H}\cdot d\underset{\sim}{l} = \int_S \underset{\sim}{J}\cdot d\underset{\sim}{S} \Rightarrow 2\pi r H_{in} = NI \Rightarrow B_{in} = \frac{\mu_r\mu_0 NI}{2\pi r}$$

- If we introduce an air gap of width l, we can model it as being a different material ($\perp$ to B field)
↳ from B.C, $\mu_0 H_{gap} = \mu_r\mu_0 H_{in}$
↳ then from Ampere's law:

$$\oint \underset{\sim}{H}\cdot d\underset{\sim}{l} = (2\pi r - l)H_{in} + l H_{gap} = NI$$

$$\Rightarrow B_{gap} = \frac{\mu_r\mu_0 NI}{2\pi r + (\mu_r - 1)l} \approx \frac{\mu_0 NI}{l} \quad \text{for } \mu l \gg 2\pi r$$

↳ the gap makes the biggest contribution to the integral because $H_{gap} = \mu_r H_{in} \Rightarrow H_{gap} >> H_{in}$ for a material with very high permeability.

• The magnetic flux across the gap depends on the cross-sectional area of the torus:

$$\phi = \int_S \underset{\sim}{B}\cdot d\underset{\sim}{S} = \frac{A\mu_0 NI}{L}$$

Electromagnetic Induction

• Faraday's law states that a time changing magnetic flux induces an e.m.f proportional to the rate of change of flux.

$$\mathcal{E} \equiv \oint \underset{\sim}{E}\cdot d\underset{\sim}{l} = -\frac{d}{dt}\int_S \underset{\sim}{B}\cdot d\underset{\sim}{S} \quad \left(= \frac{d\phi}{dt}\right)$$

↳ the e.m.f promotes a current flow whose magnetic field opposes the change that caused it (Lenz's law)

↳ using Stokes' theorem, we can write

$$\boxed{\nabla \times \underset{\sim}{E} = -\frac{\partial B}{\partial t}}$$

Self-inductance

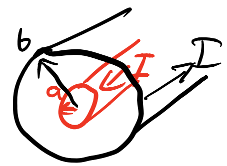
• Self-inductance is the flux linked back to a circuit as a consequence of unit current flowing in the circuit

$$L \equiv \frac{\phi}{I}$$ ← dependent on geometry

e.g for a solenoid, $B_{in} = \mu_0 n I$, $\phi = B_{in} A \cdot nL$

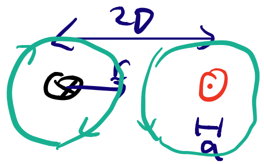
$$\therefore L = \frac{\Phi}{I} = \frac{\mu_0 n^2 LAI}{I} = \mu_0 n^2 LA$$

e.g for a coaxial cable, $B(r) = \frac{\mu_0 I}{2\pi r}$ $a<r<b$



$$\phi = L\int_a^b B(r)\,dr \Rightarrow L = \frac{\mu_0 L}{2\pi}\ln(b/a)$$

e.g for a pair of narrow wires, we first analyse one wire. $B(r) = \frac{\mu_0 I}{2\pi r}$



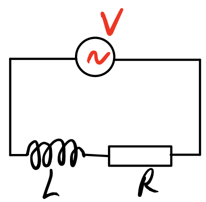
$\therefore \phi_1/l = \int_a^{2D-a} \frac{\mu_0 I}{2\pi r} dr$

$\therefore \frac{L}{l} = \frac{(\phi_1+\phi_2)/l}{I} = \frac{\mu_0}{\pi}\ln\left(\frac{2D}{a}\right)$

· Self-inductance relates the voltage across a circuit to the rate of change of current. Across some small gap in the circuit $V_{gap} = \frac{d\phi}{dt} = L\frac{dI}{dt}$

↳ hence breaking a circuit with high L can create a huge voltage (since $I \to 0$ very quickly).

· Inductors store energy. In an RL circuit, we have $V = IR + L\dot{I}$



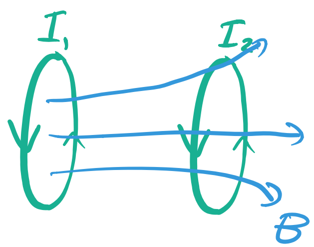
$\therefore P = VI = I^2R + \frac{d}{dt}\left(\frac{1}{2}LI^2\right)$

↳ hence the energy stored is $\boxed{U_L = \frac{1}{2}LI^2}$

↳ similarly for a capacitor: $\boxed{U_C = \frac{1}{2}CV^2}$

↳ i.e inductors store in B-field, capacitors in E-field.

· A circuit can induce an emf in another circuit. The mutual inductance is defined by $M_{21} = \Phi_2/I_1$. In fact, this quantity is symmetric. Proof:



↳ consider $I_1 = I_2 = 0$ initially then gradually increment I_1

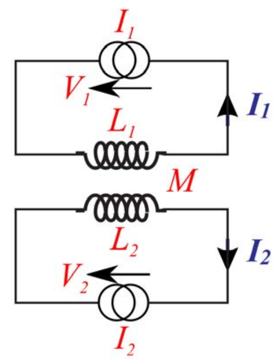
↳ the energy in the B-field is then $U_1 = \frac{1}{2}L_1I_1^2$

↳ if I_2 is now incremented, it creates $U_2 = \frac{1}{2}L_2I_2^2$ but also induces an emf in the first coil

$\therefore U = \frac{1}{2}L_1I_1^2 + \frac{1}{2}L_2I_2^2 + M_{12}I_1I_2$

↳ this must be the same if I_2 was incremented first $\therefore M_{12} = M_{21}$.

· For a simple coupled circuit:



$V_1 = L_1\frac{dI_1}{dt} + M\frac{dI_2}{dt}$

$V_2 = L_2\frac{dI_2}{dt} + M\frac{dI_1}{dt}$

↳ the power of the circuit is

$P = V_1I_1 + V_2I_2$

$= \frac{d}{dt}\left[\frac{1}{2}I_1^2L_1 + \frac{1}{2}I_2^2L_2 + I_1I_2M\right]$

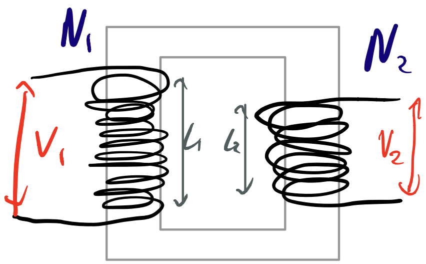
↳ hence the circuit energy can be derived by circuit analysis.

· We can show that $L_1L_2 \geq M^2$, or equivalently define a coupling coefficient $M = k\sqrt{L_1L_2}$, $0 \leq k \leq 1$

↳ $k=1$ means perfect coupling, e.g a doubly wound solenoid.

↳ essentially a constant times the geometric mean

Transformers



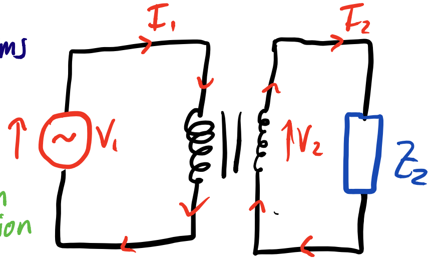
- For an ideal transformer, the same flux passes through both cores, i.e the coupling constant is unity

$$\Rightarrow \quad V_2/V_1 = N_2/N_1$$

- Using the expression for the self-inductance of a coil:

$$\frac{L_1}{L_2} = \left(\frac{N_1}{N_2}\right)^2 \frac{l_2}{l_1}$$

- We can analyse how impedance transforms in a transformer circuit, with a sign convention as shown.



$$\phi = LI_1 - MI_2$$ ← because I_2 in opposite direction

↳ using $V(t) = Ve^{i\omega t}$:

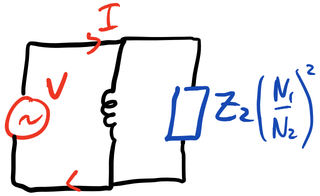
$$V_1 = L_1 \frac{dI_1}{dt} - M\frac{dI_2}{dt} = i\omega L_1 I_1 - i\omega M I_2$$

$$V_2 = -L_2 \frac{dI_2}{dt} + M\frac{dI_1}{dt} = -i\omega L_2 I_2 + i\omega M I_1$$

↳ then sub $V_2 = I_2 Z_2$, eliminate I_2, and rewrite in terms of transformer dimensions.

$$\Rightarrow \quad Z_1 = \frac{i\omega L_1 Z_2 (N_1/N_2)^2}{i\omega L_1 + Z_2 (N_1/N_2)^2}$$

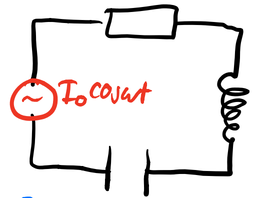
↳ i.e the input impedance is equivalent to $i\omega L_1$ in parallel with a scaled Z_2



↳ generally, $\omega L_1 >> Z_2 (N_1/N_2)^2$

$$\Rightarrow \quad Z_1 \approx Z_2 (N_1/N_2)^2$$

Energy flow in resonant circuits



- For an RLC circuit driven with an oscillating current as input:

$$V_R(t) = IR = I_0 R\cos\omega t \quad \Rightarrow \quad P = I_0^2 R\cos^2\omega t$$

$$V_L(t) = L\dot{I} = -I_0\omega L\sin\omega t \quad \Rightarrow \quad P = -I_0^2 \omega L \cdot \tfrac{1}{2}\sin 2\omega t$$

$$V_C(t) = \frac{1}{C}\int I\,dt = I_0 \frac{1}{\omega C}\sin\omega t \quad \Rightarrow \quad P = I_0^2 \frac{1}{\omega C} \cdot \tfrac{1}{2}\sin 2\omega t$$

↳ energy is always dissipated by the resistor

↳ energy flow in/out to the magnetic field corresponds to energy flow out/in to the electric field.

- At resonance, the magnetic/electric energy storage perfectly balances, i.e energy sloshes back and forth between components.

$$\omega L = \frac{1}{\omega C} \quad \Rightarrow \quad \omega_0 = 1/\sqrt{LC}$$

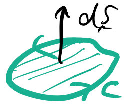
Magnetic energy

- For a single circuit, $\boxed{U = \tfrac{1}{2}LI^2 = \tfrac{1}{2}\phi I}$
- For two circuits, we can distribute the $I_1 I_2 M$ term equally between circuits and write $U = \tfrac{1}{2}\phi_1 I_1 + \tfrac{1}{2}\phi_2 I_2$
- For a collection of current loops: $\boxed{U = \frac{1}{2}\sum_{i=1}^{N} \phi_i I_i}$

ϕ_i is the total flux in loop i

* ↳ NB: this includes the self-energies of the loops, i.e different to electrostatics.

↳ it would be the same as the electrostatics case iff ϕ_i were defined to be flux due to currents in the other $N-1$ loops.



- For one of these loops:

$$\Phi_i = \int_i \underline{B}\cdot d\underline{S}_i = \oint_i \underline{A}\cdot d\underline{l}$$

$$\therefore U = \sum_i \frac{1}{2}\left(\oint I_i \underline{A}\cdot d\underline{l}\right)$$

↳ this can be generalised to a current in a volume:

$$U = \frac{1}{2}\int_V \underline{J}\cdot\underline{A}\, d\tau \quad \leftarrow \text{analogous to } U = \frac{1}{2}\int \rho V d\tau$$

- Since $\nabla\times\underline{H} = \underline{J}$, $U = \frac{1}{2}\int_V \underline{A}\cdot(\nabla\times\underline{H})\,d\tau$
- We can then use a vector identity to expand this and reason about the rate of decay of the quantities as the surface goes to infinity (same as electrostatics)

$$\therefore \boxed{U_m(\underline{r}) = \frac{1}{2\mu_0}|\underline{B}(\underline{r})|^2 = \frac{1}{2}\underline{B}(\underline{r})\cdot\underline{H}(\underline{r})}$$

energy density

Electromagnetic Waves

- $\nabla\times\underline{H} = \underline{J}$ is inconsistent with charge cons, which states that $\nabla\cdot\underline{J} = -\frac{\partial\rho}{\partial t}$. Hence we must add a displacement current. Maxwell's equations are then:

$$\boxed{\begin{aligned} \nabla\cdot\underline{D} &= \rho \\ \nabla\cdot\underline{B} &= 0 \\ \nabla\times\underline{E} &= -\dot{\underline{B}} \\ \nabla\times\underline{H} &= \underline{J} + \dot{\underline{D}} \end{aligned}}$$

→ Gauss's theorem — charge causes E field
→ Gauss's theorem — no magnetic monopoles
→ Faraday's law — E and B are coupled
→ Ampere's law — current causes B field

- In free space, $\rho = 0$ $\underline{J} = 0$ $\underline{D} = \varepsilon_0\underline{E}$ $\underline{B} = \mu_0\underline{H}$. Thus $\nabla\times\underline{E} = -\mu_0\dot{\underline{H}}$. If we take the curl again:

$$\nabla\times(\nabla\times\underline{E}) = -\mu_0\frac{\partial\,\nabla\times\underline{H}}{\partial t} \implies \boxed{\nabla^2\underline{E} = \mu_0\varepsilon_0\frac{\partial\underline{E}}{\partial t^2}}$$

↳ an identical equation can be derived for $\nabla^2\underline{H}$

↳ this is a wave equation because we can separate $\nabla^2\underline{E} = (\nabla^2 E_x, \nabla^2 E_y, \nabla^2 E_z)$. Thus $c = 1/\sqrt{\mu_0\varepsilon_0}$

- We can look for plane waves propagating in the z direction, where $\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0$.

$$\nabla\times\underline{H} = \left(-\frac{\partial H_y}{\partial z}, \frac{\partial H_x}{\partial z}, 0\right) = \varepsilon_0\frac{\partial\underline{E}}{\partial t}$$

↳ $\partial E_z/\partial t = 0$ tells us that EM waves are purely transverse

↳ there are two orthogonal polarisations (characterised by the axis of E), each with an E-H pair.

↳ both the E_x and H_y are transverse waves with speed c.

↳ For a wave in a dielectric or magnetic material, the refractive index is $\boxed{n = \sqrt{\varepsilon_r \mu_r}}$

- We can use the relationship between the space-derivative of H_y and time-derivative of E_x, along with the form of a plane wave $E_x = \mathrm{Re}\left[E_0 \exp(i(\omega t - kx))\right]$ to show: $\dfrac{E_x}{H_y} = \dfrac{k}{\varepsilon_0 \omega} = \sqrt{\dfrac{\mu_0}{\varepsilon_0}} \approx 377\,\Omega$

  ↳ this quantity is the impedance of free space Z_0

  ↳ for propagation in some medium, $Z = \sqrt{\mu_r\mu_0 / \varepsilon_r \varepsilon_0}$

  ↳ Z can be thought of as quantifying the 'response' (H) as a result of a 'disturbance' (E).

- For a general plane wave:

$$\underset{\sim}{E}(\underset{\sim}{x}, t) = \underset{\sim}{E_0} \exp[i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)]$$
$$\underset{\sim}{H}(\underset{\sim}{x}, t) = \underset{\sim}{H_0} \exp[i(\underset{\sim}{k} \cdot \underset{\sim}{x} - \omega t)]$$

} remember to take real parts

↳ it is easy to show $\nabla \cdot \underset{\sim}{E} = i \underset{\sim}{k} \cdot \underset{\sim}{E}$, $\nabla \times \underset{\sim}{E} = i \underset{\sim}{k} \times \underset{\sim}{E}$ (likewise for $\underset{\sim}{H}$)

↳ thus: $\underset{\sim}{k} \times \underset{\sim}{E_0} = \omega \mu_0 \underset{\sim}{H_0}$, $\underset{\sim}{k} \times \underset{\sim}{H_0} = -\omega \varepsilon_0 \underset{\sim}{E_0}$ } $\underset{\sim}{E}, \underset{\sim}{H}, \underset{\sim}{k}$ form a right-handed system

↳ also, $\boxed{\underset{\sim}{H_0} = \frac{1}{Z_0} \hat{\underset{\sim}{k}} \times \underset{\sim}{E_0}}$ → more general than $|E| = Z|H|$

- Because of Fourier theory, any field can be described as some (possibly infinite) series of plane waves

$$E(\underset{\sim}{x}, t) = \iiiint \underset{\sim}{A_s}(\underset{\sim}{k}, \omega) e^{ik_x x} e^{ik_y y} e^{ik_z z} e^{-i\omega t} \, d^3k \, d\omega$$

↳ $\underset{\sim}{A_s}$ is the spectral function

↳ note that the K's are not independent

Energy flow

- The rate at which work is done by a field on a charge is:

$$P = \frac{d}{dt}(q \underset{\sim}{E} \cdot d\underset{\sim}{l}) = q \underset{\sim}{E} \cdot \underset{\sim}{v}$$

↳ for a charge distribution: $\boxed{P = \int_V \underset{\sim}{E} \cdot \underset{\sim}{J} \, dV}$

* ↳ hence the power dissipated by a current per unit volume is: $P/V = \underset{\sim}{E} \cdot \underset{\sim}{J}$

- To calculate the power flux, we analyse the quantity:

$$\nabla \cdot (\underset{\sim}{E} \times \underset{\sim}{H}) = \underset{\sim}{H} \cdot (\nabla \times \underset{\sim}{E}) - \underset{\sim}{E} \cdot (\nabla \times \underset{\sim}{H})$$

↳ by substituting from Maxwell and taking the volume integral (using div. theorem on LHS):

$$-\oint_S \underbrace{(\underset{\sim}{E} \times \underset{\sim}{H})}_{\text{defines propagation}} \cdot d\underset{\sim}{S} = \int_V \left(\underbrace{\frac{\partial}{\partial t}\left[\tfrac{1}{2}\mu_0 \underset{\sim}{H} \cdot \underset{\sim}{H}\right] + \frac{\partial}{\partial t}\left[\tfrac{1}{2}\varepsilon_0 \underset{\sim}{E} \cdot \underset{\sim}{E}\right]}_{\text{rate of increase in stored energy}} + \underbrace{\underset{\sim}{E} \cdot \underset{\sim}{J}}_{\text{energy dissipated}} \right) dV$$

- Thus since the RHS is the change in power, $\underset{\sim}{E} \times \underset{\sim}{H}$ must be the power flux into the volume.

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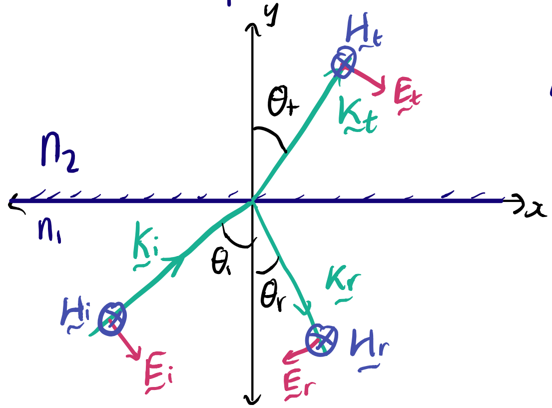
- This defines the Poynting vector: $\boxed{\underline{N} = \underline{E} \times \underline{H}}$
  - ↳ quantifies the direction and magnitude of the power flux
  - ↳ nonlinear in field so cannot be superposed.
  - ↳ e.g for a plane wave travelling in $+z$, $\underline{E}=(E_x,0,0)$, $\underline{H}=(0,H_y,0) \Rightarrow \underline{N}=(0,0,E_x^2/Z)$
- If we are instead using complex vector fields $\hat{\underline{E}}$ and $\hat{\underline{H}}$, the average power flux is: $\boxed{\left|\frac{1}{2}\mathrm{Re}[\hat{\underline{E}} \times \hat{\underline{H}}^*]\right|}$

  magnitude of the vector
  - ↳ the max rate at which energy sloshes back and forth is given by: $\left|\frac{1}{2}\mathrm{Im}[\hat{\underline{E}} \times \hat{\underline{H}}^*]\right|$
  - ↳ when $\underline{E}$ and $\underline{H}$ are in phase, the complex power is a real quantity
- For a wave normally incident on an absorbing surface, the energy density is given by $\frac{\text{power}}{\text{volume}} \times 1$ second

  $\therefore U = \frac{|\underline{N}| \cdot A}{V} = |\underline{N}| \cdot A / Acdt = |\underline{N}|/c$
  - ↳ for a photon, $E = pc \Rightarrow U = |\underline{g}|c$ where $\underline{g}$ is the radiation momentum density
  - ↳ hence $\underline{g} = \underline{N}/c^2$, and $d\underline{p} = \underline{g}Acdt$
- The radiation pressure is the rate of change of momentum per unit area. $\underline{R} = \underline{g}c \Rightarrow \boxed{\underline{R} = \underline{N}/c}$
  - ↳ $|\underline{R}|$ doubles if the surface reflects radiation.

Reflection and transmission

- Consider a plane wave incident on a plane dielectric boundary



- ↳ incident waved polarised in plane of page
- ↳ no assumptions made on directions or magnitudes of reflection/transmission.

- On the x-axis, the parallel components of the fields must be continuous:

$$E_{i0}\exp[i(k_i x\sin\theta_i - \omega_i t)]\cos\theta_i - E_{r0}\exp[i(k_r x\sin\theta_r - \omega_r t)]\cos\theta_r = E_{t0}\exp[i(k_t x\sin\theta_t - \omega_t t)]\cos\theta_t$$

  - ↳ this can only be true in general if their phases match. Thus: $\omega_i = \omega_r = \omega_t$

    $k_i\sin\theta_i = k_r\sin\theta_r = k_t\sin\theta_t$
  - ↳ but k depends on n via $k = n\omega/c$
  - ↳ hence: $\theta_i = \theta_r$ ← law of reflection

    $n_1\sin\theta_i = n_2\sin\theta_t$ ← Snell's law.
- We can then analyse the power transmitted and reflected

  $(E_{i0} - E_{r0})\cos\theta_i = E_{t0}\cos\theta_t$
  - ↳ then match $H_\parallel$: $H_{i0} + H_{r0} = H_{t0}$

    $\Rightarrow n_1(E_{i0} + E_{r0}) = n_2 E_{t0}$ $\quad \frac{E}{H} = \frac{Z_0}{n}$

↳ these 2 eqs can be solved for 2 unknowns:

$$r_{\parallel} = \frac{E_{r0}}{E_{i0}} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

$$t_{\parallel} = \frac{E_{t0}}{E_{i0}} = \frac{2\cos\theta_i}{(n_2/n_1)\cos\theta_i + \theta_t}$$

- This assumed that the $\underset{\sim}{E}$ field was polarised along the plane of incidence. We can instead derive it for the case that $\underset{\sim}{E}$ is polarised perpendicular to the page:

$$r_{\perp} = \frac{E_{r0}}{E_{i0}} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

$$t_{\perp} = \frac{E_{t0}}{E_{i0}} = \frac{2\cos\theta_i}{\cos\theta_i + (n_2/n_1)\cos\theta_t}$$

- These equations are Fresnel's relations, and explain a number of optical phenomena.
- For normally incident light, we can use a small-angle approximation to find the power reflection coefficient:

$$R_{\parallel} = |r_{\parallel}|^2 = \left(\frac{n-1}{n+1}\right)^2 \quad \leftarrow \text{same for } R_{\perp},\ r_{\perp}$$

- There is a particular angle for which $r_{\parallel} = 0$. This is the Brewster angle: $\tan\theta_B = n_2/n_1$
- If $n_1 > n_2$ (e.g. glass → air), there is a critical angle beyond which total internal reflection occurs. $\sin\theta_c = n_1/n_2$

↳ an evanescent wave is produced, travelling along the surface.

Waves in plasmas

- A plasma is a region of space where free electrons and their parent ions are present. We assume that ions are stationary, thus the electrons obey:

$$m_e \frac{d^2 \underset{\sim}{r}}{dt^2} = -e(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$$

↳ electrons in plasmas are not fast, and we know that for a plane wave $E_x = cB_y$. Hence $\underset{\sim}{v} \times \underset{\sim}{B}$ can be ignored. Hence $\underset{\sim}{r} = \frac{e}{m_e\omega^2}\underset{\sim}{E_0}\exp[i(kz-\omega t)]$

↳ i.e. electrons oscillate around their ions with amplitude inversely proportional to the radiation freq.

- The separation of electron from ion creates a dipole:

$$\underset{\sim}{p} = -e\underset{\sim}{r} = -\frac{e^2}{m_e\omega^2}\underset{\sim}{E}$$

↳ we can then analyse the relationship between the electric field and the polarisation to derive the plasma's permittivity:

$$\underset{\sim}{P} = \underbrace{n_v}_{\text{number density}} \underset{\sim}{p} = -\frac{n_v e^2}{m_e\omega^2}\underset{\sim}{E} \quad \text{and} \quad \underset{\sim}{P} = \varepsilon_0(\varepsilon_r - 1)\underset{\sim}{E}$$

$$\Rightarrow \boxed{\varepsilon_r = 1 - \frac{\omega_p^2}{\omega^2}, \quad \omega_p^2 = \frac{n_v e^2}{\varepsilon_0 m_e}}$$

↳ ω_p is the plasma frequency, and is a material's property.

↳ using the dielectric formalism allows us to analyse behaviour in the same way as we would an insulator.

- The refractive index of a dielectric is $n = \sqrt{\varepsilon_r \mu_r}$ hence for a plasma: $n = \sqrt{\varepsilon_r} = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$
- The refractive index is imaginary when the radiation freq is below the plasma frequency.

$$n = i\beta \Rightarrow k = \frac{n\omega}{c} = \frac{i\beta}{c}$$

$$\therefore \ E = E_0 \exp\left[-\frac{\omega\beta z}{c}\right] \exp[-i\omega t]$$

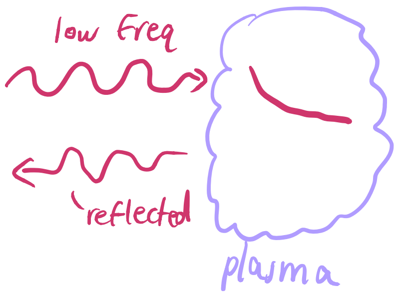
↳ i.e there is a non-propagating evanescent wave

↳ the magnetic field is:

$$H_y = \sqrt{\frac{\varepsilon_r \varepsilon_0}{\mu_0}} = \frac{i\beta}{Z_0} E_x$$

↳ because E_x and H_y are out of phase, the average power transmitted is zero.

↳ hence all energy must be reflected.



- Above the plasma frequency, the travelling waves are dispersive:

$$V_p = \frac{\omega}{k} = \frac{c}{n} = c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{-1/2}$$

$$V_g = \frac{d\omega}{dk} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

↳ while $v_p > c$, it is the group velocity that carries information and $V_g < c$.

↳ $V_g V_p = c^2$, which applies to waveguides also.

Waves in conducting media

- From Maxwell's equations:

$$\nabla \times \underset{\sim}{H} = J + \frac{\partial D}{\partial t} = \sigma \underset{\sim}{E} + \varepsilon_r \varepsilon_0 \frac{\partial E}{\partial t}$$

↳ assuming plane waves for $\underset{\sim}{E}$ and $\underset{\sim}{H}$:

$$\nabla \times \underset{\sim}{H} = -i\omega\varepsilon_0\left(\varepsilon_r + \frac{i\sigma}{\omega\varepsilon_0}\right)\underset{\sim}{E}$$

↳ we can thus model a real conductor as a dielectric with constant: $\boxed{\varepsilon_r' = \left(\varepsilon_r + \frac{i\sigma}{\omega\varepsilon_0}\right)}$

- In a good conductor, the imaginary part is much greater so $\varepsilon_r' \simeq \frac{i\sigma}{\omega\varepsilon_0}$. The refractive index is thus complex: $n = \sqrt{\varepsilon_r \mu_r} = \pm(1+i)\sqrt{\frac{\sigma\mu_r}{2\omega\varepsilon_0}}$

↳ with $k = \frac{\omega}{c/n} = \frac{\omega}{n}\sqrt{\mu_0\varepsilon_0}$, the solution for $\underset{\sim}{E}$ is:

$$\underset{\sim}{E} = \underset{\sim}{E_0} \exp\left[-\frac{z}{\delta}\right] \exp\left[i\left(\frac{z}{\delta} - \omega t\right)\right]$$

where $\boxed{\delta = \sqrt{\frac{2}{\sigma\omega\mu_0\mu_r}}}$ is the skin depth

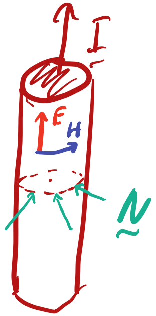
↳ the skin depth thus characterises the amplitude decay of the travelling wave in a conductor.

- A conductor introduces a $\pi/4$ phase difference between $\underset{\sim}{E}$, $\underset{\sim}{H}$

↳ impedance is now complex

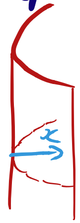
↳ power is dissipated in the conductor.

- Consider a wire carrying a current with freq ω
  - ↳ because of the finite conductivity of the material, there is an E field on the surface. Combined with the B field from the wire, there is $\underline{N}$ pointing radially inwards



  - ↳ this energy flow must decay based on the skin depth
  - ↳ we define a new coordinate $x = a - r$:

$$J_z = J_0 \exp\left[-\frac{x}{\delta}\right] \exp\left[i\left(\frac{x}{\delta} - \omega t\right)\right]$$



- Hence alternating currents tend to flow on the surface of the wire — this is the skin effect
  - ↳ because the current is confined in a smaller region, the resistance of the wire increases as freq ↑.

$$\hat{I} = \int J_z dS \approx 2\pi a \int J_z(x)\, dx$$

integrate to ∞ because of exponential decay

$$\therefore \hat{I} \approx 2\pi a J_0 \exp[-i\omega t] \int_0^\infty \exp\left[\frac{x}{\delta}(i-1)\right] dx$$

$$\Rightarrow \hat{I} = \pi a J_0 \delta (1+i) e^{-i\omega t}$$

$$\Rightarrow \langle I(t) \rangle^2 = (\pi a J_0 \delta)^2$$

  - ↳ similarly, $\langle J_z(t) \rangle^2 = \frac{1}{2} J_0^2 \exp\left[-\frac{2z}{\delta}\right]$
  - ↳ $dP = I^2 R = \frac{J^2 dA}{\sigma} L$ and P is found by integrating
  - ↳ $R = P / \langle I(t) \rangle^2 = \boxed{\frac{1}{2\pi a \sigma \delta}}$
- The resistance per unit length is as if all current flowed uniformly in a shell of thickness δ

Guided waves

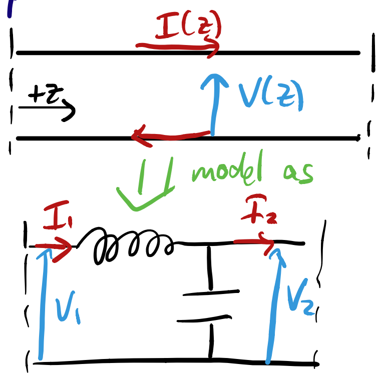
- For long wires ($d > \lambda$), we must take into account the fact that V and I have position-dependence (since they are the result of EM waves).

d is the dimension of the circuit

- Transmission lines can be used for $d \approx \lambda$, while for $\lambda \ll d$ we need waveguides.

Transmission lines

- Wires not only conduct current, they guide EM energy.
- The simplest transmission line setup is a pair of wires
  - ↳ using L and C per unit length:



$$dV = V_2 - V_1 = -(L dz) \frac{\partial I}{\partial t}$$

$$dI = I_2 - I_1 = -(C dz) \frac{\partial V}{\partial t}$$

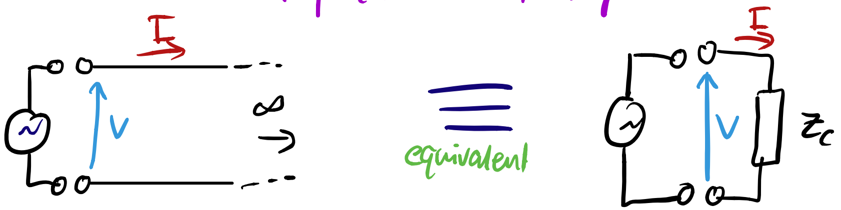
  - ↳ in the limit $dz \to 0$:

$$\frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} \qquad \frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t}$$

$$\Rightarrow \frac{\partial^2 V}{\partial t^2} = \frac{1}{LC} \frac{\partial^2 V}{\partial z^2} \qquad \frac{\partial^2 I}{\partial t^2} = \frac{1}{LC} \frac{\partial^2 I}{\partial z^2}$$

  - ↳ hence there are voltage and current waves with $v = \pm \frac{1}{\sqrt{LC}}$
  - ↳ with the engineering convention $V = V_0 \exp[i(\omega t - kz)]$, we have $kV = \omega L I \Rightarrow Z = \frac{V}{I} = \sqrt{L/C}$
  - ↳ this is the characteristic impedance of the line, Z_c

- We could replace the transmission line with a load of impedance $Z = Z_c$ without changing the behaviour at the terminals $\Rightarrow$ impedance matching.

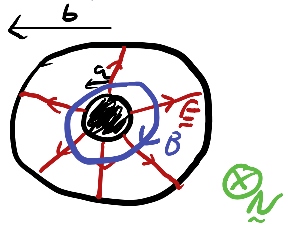


- For the pair of wires, $C = \frac{\pi \varepsilon_0}{\ln(2D/a)}$ $\quad L = \frac{\mu_0}{\pi}\ln(2D/a)$

$$\Rightarrow \quad Z = Z_0 \frac{\ln(2D/a)}{\pi}$$

↳ if we fill the space between them with a dielectric, $Z' = \frac{Z}{n}$

- For a coaxial cable, $C = \frac{2\pi\varepsilon_0}{\ln(b/a)}$ $\quad L = \frac{\mu_0}{2\pi}\ln(b/a)$

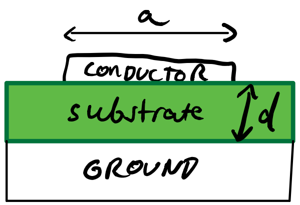


$$\Rightarrow Z = Z_0 \frac{\ln(b/a)}{2\pi}$$

↳ most cables are manufactured with $Z = 50\Omega$ or $Z = 75\Omega$

↳ the cable may be partially filled with dielectrics, but will only support a transverse EM wave if there is radial symmetry (i.e cylinders).

- On PCBs, a useful setup is the strip transmission line. For $d \ll a$: ignore edge effects



$C = \frac{\varepsilon_r \varepsilon_0 a}{d}$, $\quad L = \frac{\mu_0 d}{a}$ $\quad \Rightarrow \quad Z = \frac{Z_0}{n}\frac{d}{a}$

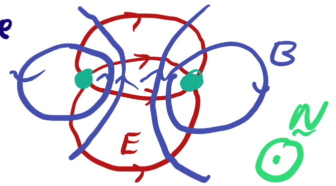
↳ hence Z can be controlled by changing the width of the conductor.

↳ in reality we would use more complicated equations that describe edge effects.

↳ technically a TEM wave is not supported since the dielectric only fills half of space, but it is a good approx if λ much bigger than the dimensions of the circuit a and d (i.e low freqs).

Power flow on transmission lines

- For most setups, B and E are everywhere perpendicular, so the Poynting vector always points along the wire.



- We can quantify the power as the negative of the rate of change of stored energy:

$$-W = -\int_a^b \tfrac{1}{2}LI^2 + \tfrac{1}{2}CV^2\, dz$$

$$\Rightarrow -\frac{dW}{dt} = -\int_a^b LI\frac{\partial I}{\partial t} + CV\frac{\partial V}{\partial t}\, dz$$

using transmission-line equations

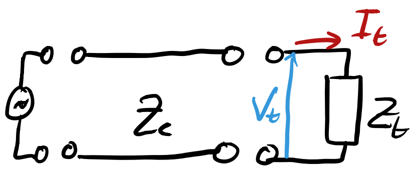
$$= \int_a^b I\frac{\partial V}{\partial z} + V\frac{\partial I}{\partial z}\, dz$$

$$= [IV]_b - [IV]_a$$

↳ hence $P = VI$ as expected.

- If we terminate a transmission line with a load that matches Z_c, the load absorbs all the power.

- If the line is terminated with an impedance that doesn't match Z_c, some power will be reflected.



$$V_i = V_1 \exp[j(\omega t - kz)] \qquad V_r = V_2 \exp[j(\omega t + kz)]$$
$$I_i = I_1 \exp[j(\omega t - kz)] \qquad I_r = I_2 \exp[j(\omega t + kz)]$$
incident reflected

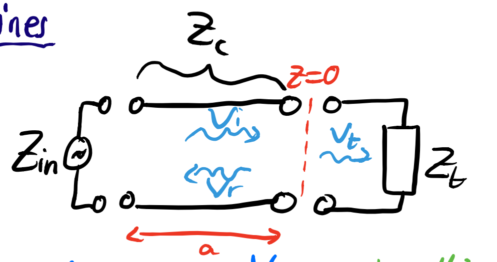
↳ matching B.Cs,
$$V_t = V_i + V_r$$
$$I_t = I_i + I_r$$
$$Z_t = V_t / I_t$$
place origin ($z=0$) wherever easiest. In this case, at load

↳ this gives us the voltage reflection coefficient
$$r_v = \frac{V_1}{V_2} = \frac{Z_t - Z}{Z_t + Z}$$

↳ we can find the current reflection coefficient to get the total power.

Input Impedances of transmission lines

- Consider a mismatched line with $Z_{in} = Z_c$ such that the source absorbs all reflections



$$Z_{in} = \left.\frac{V_i + V_r}{I_i + I_r}\right|_{z=-a}, \quad I_i = \frac{V_i}{Z_c}, \quad I_r = \frac{V_r}{-Z_c}$$
travelling backwards

$$\therefore \frac{Z_{in}}{Z_c} = \frac{Z_t \cos ka + iZ_c \sin ka}{Z_c \cos ka + i Z_t \sin ka}$$

↳ hence the dimension of the circuit defines the response

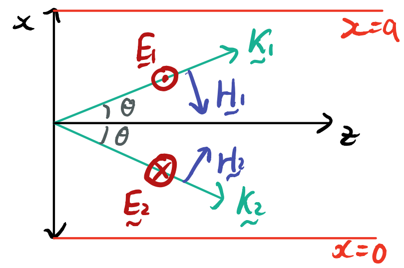
- For a shorted line, $Z_t = 0 \Rightarrow Z_{in}/Z_c = i \tan ka$
  ↳ purely imaginary since the load cannot absorb power
  ↳ thus a shorted wire can be used to synthesise an impedance.
  ↳ similarly, for an open-circuited line $Z_t \to \infty \Rightarrow Z_{in}/Z_c = -i \cot ka$
- For the special case of a quarter-wavelength line, $\cos ka = 0$

$$\therefore \frac{Z_{in}}{Z_c} = \frac{Z_c}{Z_t} \Rightarrow \boxed{Z_{in} = \frac{Z_c^2}{Z_t}}$$

↳ hence the $\lambda/4$ line ensures there is no reflection
↳ this only works for a single frequency

Waveguides

- Losses in transmission lines increase significantly at very high freqs due to the skin effect.
- Waveguides allow EM waves to be propagated through hollow tubes, without the need for a second conductor.
  ↳ transmission lines are simple special cases of waveguides that can be analysed in terms of V and I rather than $\underline{E}$ and $\underline{H}$.
  ↳ we consider two plane waves with wavevectors $\underline{k}_1, \underline{k}_2$ travelling at angles $\pm\theta$ to the z-axis.



- By considering the resultant E field:

$$E_y = E_0\left(\exp[ik_1\cdot r] - \exp[ik_2\cdot r]\right)e^{-i\omega t}$$
$$= E_0\left(\exp[i(k x\sin\theta + k z\cos\theta)] - \exp[i(-k x\sin\theta + k z\cos\theta)]\right)e^{-i\omega t}$$
$$\therefore E_y = E_0\exp[i(kz\cos\theta - \omega t)]\cdot 2i\sin(kx\sin\theta).$$

↳ we can then fit the B.Cs of conducting plates at $x=0, x=a$, hence $Ka\sin\theta = m\pi \Rightarrow \boxed{K_x = \frac{m\pi}{a}}$, $m\in\mathbb{Z}$

↳ hence there is a standing wave between the plates and a propagating wave in the $+z$ direction.

↳ this solution also fits the B.Cs for the H field: $H_\perp = 0$ inside and just outside the conductor.

↳ there may be a nonzero $H_{\parallel}$, but if there is no field inside the conductor, this B.C can only be satisfied by a sheet of current (by Ampere's law)

↳ hence current will flow in a waveguide, which decay into the conductor via the skin effect.

- The effective wavevector for propagation is $K_g = K\cos\theta$, so the phase velocity is: $V_\phi = \frac{\omega}{K_g} = \frac{\omega}{K\cos\theta} = \frac{c}{\cos\theta}$

↳ $V_\phi \geq c$, but this is fine since only the group velocity propagates information

- Because of the vertical standing wave, only certain frequencies of the propagating wave are possible.

$$K^2 = K_x^2 + K_g^2 \Rightarrow \boxed{K_g^2 = K^2 - \frac{m^2\pi^2}{a^2}}$$

- For $a < \frac{\lambda}{2}$, the standing wave cannot be satisfied

↳ this could be a good thing, e.g for a two-strip transmission line, waveguide behaviour is not desired.

↳ for high freqs, the only solution is to make the lines very small.

- We can now introduce conducting plates in the y direction

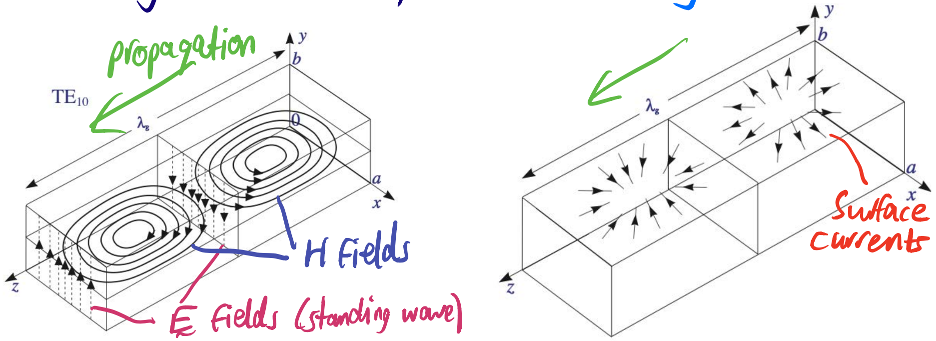
↳ since E is in the y direction, $E_{\parallel} = 0$ as required

↳ $H_y = 0$ so the B.C for H is automatically satisfied.

- We then have a rectangular waveguide, which supports a transverse electric (TE) wave.

↳ the H field is not transverse, so this is not a TEM wave (unlike for transmission lines).

↳ the lowest TE mode is TE_{10}, i.e lowest order standing wave in x, no variation in y.



↳ we can make cuts in the waveguide walls to introduce components, but they must not prevent current flow.

- The solution for a general TE_{mn} mode is:

$$E_x = A_0 k_y \cos(k_x x)\sin(k_y y)\cos(k_z z - \omega t)$$
$$E_y = -A_0 k_x \sin(k_x x)\cos(k_y y)\cos(k_z z - \omega t)$$
$$E_z = 0$$

with $(k_x, k_y, k_z) = \left(\frac{m\pi}{a}, \frac{n\pi}{b}, k_g\right)$, $m, n \in \mathbb{Z}$

- Since $|\underset{\sim}{k}|^2 = \frac{\omega^2}{c^2}$ by definition:

$$\boxed{k_g^2 = \frac{\omega^2}{c^2} - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}}$$

↳ for a propagating wave, $k_g^2 > 0$, hence the cutoff frequency is $f_c = c\sqrt{\frac{m^2}{4a^2} + \frac{n^2}{4b^2}}$

↳ below this there can only be evanescent waves.

Robert Andrew Martin

Summary of Important Formulae

Maxwell's Equations

Free space: | **Matter:**

$$\nabla \cdot \underset{\sim}{E} = \frac{\rho}{\varepsilon_0} \quad \xrightarrow[\nabla\cdot \underset{\sim}{P} = -\rho_b]{\underset{\sim}{D} = \varepsilon_0 \underset{\sim}{E} + \underset{\sim}{P}} \quad \nabla \cdot \underset{\sim}{D} = \rho_{free}$$

$$\nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t} \quad \xrightarrow{\text{SAME}} \quad \nabla \times \underset{\sim}{E} = -\frac{\partial \underset{\sim}{B}}{\partial t}$$

$$\nabla \cdot \underset{\sim}{B} = 0 \qquad \nabla \cdot \underset{\sim}{B} = 0$$

$$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J} + \mu_0 \varepsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t} \quad \xrightarrow[\nabla\times \underset{\sim}{M} = \underset{\sim}{J}_m]{\underset{\sim}{B} = \mu_0(\underset{\sim}{H} + \underset{\sim}{M})} \quad \nabla \times \underset{\sim}{H} = \underset{\sim}{J}_{free} + \frac{\partial \underset{\sim}{D}}{\partial t}$$

D and H

$$\underset{\sim}{D} = \varepsilon_0 \underset{\sim}{E} + \underset{\sim}{P} \qquad \underset{\sim}{P} = \varepsilon_0 \chi \underset{\sim}{E} \qquad \underset{\sim}{D} = \varepsilon_r \varepsilon_0 \underset{\sim}{E}$$

$$\underset{\sim}{H} = \frac{1}{\mu_0}\underset{\sim}{B} + \underset{\sim}{M} \qquad \underset{\sim}{M} = \chi_m \underset{\sim}{H} \qquad \underset{\sim}{B} = \mu_r \mu_0 \underset{\sim}{H}$$

Lorentz force law: $\underset{\sim}{F} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

Power:

$$U = \frac{1}{2}\int_V \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \, d\tau$$
$$\underset{\sim}{N} = \underset{\sim}{E} \times \underset{\sim}{H}$$