# Fourier Transforms

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$$\boxed{\begin{aligned} f(x) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos(k_n x) + b_n\sin(k_n x) \\ a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos(k_n x)\,dx \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin(k_n x)\,dx \end{aligned}}$$

- It is sometimes simpler to use a complex representation:

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} C_n e^{ik_n x} \qquad C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-ik_n x}\,dt}$$

- For a nonperiodic function, $T \to \infty$:
  - ↳ $k_{n+1} - k_n = \frac{2\pi}{T} \to 0$
  - ↳ $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{ik_n x} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} T C_n e^{ik_n x}\Delta k$

$$\boxed{\begin{aligned} &\text{FORWARD:} && \tilde{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx \\ &\text{INVERSE:} && f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(x)e^{ikx}\,dk \end{aligned}}$$

- Generally:
  - ↳ discontinuous $f(x) \Rightarrow$ broad $\tilde{f}(k)$
  - ↳ width of $\tilde{f}(k)$ inverse of width of $f(x)$

- If $f(x)$ is not 'well-behaved', e.g. has infinite extent, the FT may involve generalised functions. e.g. $FT[1] = 2\pi\delta(x)$.

## Properties of the FT

- Linear
- Rescaling (real $\alpha$)

$$g(x) = f(\alpha x) \Leftrightarrow \tilde{g}(k) = \frac{1}{|\alpha|}\tilde{f}\left(\frac{k}{\alpha}\right)$$

- Shift / exponential

$$g(x) = f(x-a) \Leftrightarrow \tilde{g}(k) = \tilde{f}(k)e^{-ika}$$
$$g(x) = e^{iax}f(x) \Leftrightarrow \tilde{g}(k) = \tilde{f}(k-\alpha)$$

- Differentiation:

$$g(x) = f'(x) \Rightarrow \tilde{g}(k) = ik\tilde{f}(k)$$

↳ prove using integration by parts

- Multiplication:

$$g(x) = xf(x) \Rightarrow \tilde{g}(k) = i\tilde{f}'(k)$$

↳ because $\tilde{g}(k) = \int_{-\infty}^{\infty} xf(x)e^{-ikx}dx = i\int_{-\infty}^{\infty} f(x)\frac{d}{dk}e^{-ikx}dx$

- $\boxed{\text{Duality: } g(x) = \tilde{f}(x) \Leftrightarrow \tilde{g}(k) = 2\pi f(-k)}$

↳ $\tilde{g}(k) = \int_{-\infty}^{\infty}\tilde{f}(x)e^{-ikx}dx$. Sub $x = -x$, hence inverse FT.

- Preserves symmetry: $f(-x) = \pm f(x) \Rightarrow \tilde{f}(-k) = \pm\tilde{f}(k)$

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## Convolution & Correlation

- The convolution of two functions is defined by:

$$\boxed{[f * g](x) = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi}$$

↳ it is a symmetric observation
↳ intuitively, one function gets spread out by the other
↳ can also be thought of as the area of overlap as $g$ scans the real axis

- In statistics, let $f(x)$, $g(y)$ be independent pdfs for random variables $X$, $Y$. What is the pdf of $Z = X + Y$?
  ↳ $h(z)\delta z = P(z < Z < z + \delta z)$
  ↳ for a given $x$: $h(z)\delta z | x = P(z - x < y < z + \delta z - x)$
  $= g(z-x)$
  ↳ we then integrate over all $x$
  $$h(z)\delta z = \int_{-\infty}^{\infty} f(x)\, g(z-x)\, dx = [f * g](z)$$

- The FT of a convolution:

$$\tilde{h}(k) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi \right] e^{-ikx} dx$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, e^{-ikx}\, dx\, d\xi \qquad \text{let } z = x - \xi$$
$$= \tilde{f}(k) \cdot \tilde{g}(k).$$

- The convolution theorem states that:

$$\boxed{\begin{aligned} \mathcal{F}[f * g] &= \mathcal{F}[f] \cdot \mathcal{F}[g] \\ \mathcal{F}[f(x) g(x)] &= \frac{1}{2\pi} \mathcal{F}[f] * \mathcal{F}[g] \end{aligned}}$$

- Thus we can deconvolve a measured signal by dividing in the Fourier domain, provided we know the convolution function.

- The correlation of two functions $h = f \otimes g$

$$\boxed{h(x) = \int_{-\infty}^{\infty} [f(y)]^* g(x+y)\, dy}$$

↳ $\tilde{h}(k) = [\tilde{f}(k)]^* \cdot \tilde{g}(k)$

## Power spectra

- From the definition of correlation:

$$\int_{-\infty}^{\infty} [f(y)]^* g(x+y)\, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k)\, e^{ikx} dx.$$

↳ set $x = 0$ then relabel $y \to x$ to obtain Parseval's thm:

$$\int_{-\infty}^{\infty} [f(x)]^* g(x)\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} [\tilde{f}(k)]^* \tilde{g}(k)\, dk$$

- In the special case $f = g$, the equation reduces to

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk$$

↳ $\Phi(k) = |\tilde{f}(k)|^2$ is the power spectrum of $f(x)$

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# Complex Methods

- The Fourier integrals can be treated as contour integrals in the real axis, for complex $z$ and $k$.
- Consider a general driven harmonic oscillator:

$$\ddot{x}(t) + 2\gamma\dot{x}(t) + \omega_0^2 x(t) = f(t)$$

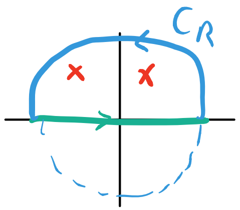
↳ equivalently to using Greens functions, take the FT:

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)\,\tilde{x}(\omega) = \tilde{f}(\omega)$$

$$\Rightarrow \quad \tilde{x}(\omega) = \tilde{g}(\omega)\tilde{f}(\omega) \quad \text{for} \quad \tilde{g}(\omega) = \frac{-1}{(\omega-\omega_+)(\omega-\omega_-)}$$

↳ $x(t) = \mathcal{F}^{-1}[\tilde{g}(\omega)\tilde{f}(\omega)] = g(t) * f(t)$

- $g(t) = \mathcal{F}^{-1}[\tilde{g}(\omega)]$ is a contour integral




↳ $g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(\omega) e^{i\omega t} d\omega$

↳ for $t>0$ we can add an upper semicircle (by Jordan's lemma this does not contribute)

$$\therefore g(t) = \frac{1}{2\pi}\oint_C \tilde{g}(\omega) e^{i\omega t} d\omega$$

- For $t<0$, we use a lower semicircle. This does not enclose any poles so $g(t) = 0$ for $t<0$ – causal behaviour
- Otherwise we can just use the residue thm to find $g(t)$

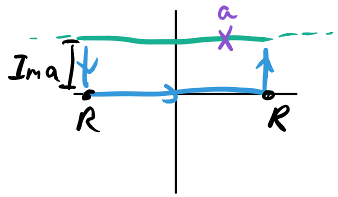
## Gaussian integration lemma

- $I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$
- This is clearly true if we shift in the real axis:

$$I' = \int_{-\infty}^{\infty} e^{-(x+a)^2} dx = \sqrt{\pi} \quad \leftarrow \text{verify with sub. } u = x-a$$

- However, it can also be shown that this holds for $a \in \mathbb{C}$

↳ let $\text{Im}\, a > 0$ WLOG. Let $C_i$ be the horizontal line with $\text{Im}\, z = \text{Im}\, a$




↳ since $e^{-z^2}$ is analytic, $\oint_c e^{-z^2} dz = 0$

↳ we thus build a rectangular contour

↳ in the limit of $R \to \infty$, this shows that

$$I = \int_{-\infty}^{\infty} e^{-(x+a)^2} dx = \sqrt{\pi}, \quad a \in \mathbb{C}$$

## The Diffusion equation

- $\frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2}$ can be solved with the FT:

$$\frac{\partial \tilde{T}(k,t)}{\partial t} = -\lambda k^2 \tilde{T}(k,t) \Rightarrow \tilde{T}(k,t) = \tilde{T}_0(k) e^{-\lambda k^2 t}$$

↳ where $\tilde{T}_0(k) = \int_{-\infty}^{\infty} e^{-ikx} T_0(x) dx$ is the initial condition

↳ using the convolution theorem

$$T(x,t) = T_0(x) * \mathcal{F}^{-1}[e^{-\lambda k^2 t}]$$

↳ $\mathcal{F}^{-1}[\dots]$ evaluated by completing the square and using the Gaussian integration lemma

- The error function $\text{erf}\, x = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2} dt$ often arises in these problems