Robert Andrew Martin

Ordinary Differential Equations

2nd order ODEs

- Generally: $y''(x) + p(x)y'(x) + q(x)y(x) = f(x)$
  - ↳ i.e $Ly = f$
  - ↳ if $f(x) = 0$, the ODE is homogeneous
- Functions $y_1(x)$ and $y_2(x)$ are linearly independent if
  $$Ay_1(x) + By_2(x) \Rightarrow A = B = 0$$
- If we can construct two linearly independent solutions to the homogeneous equation $\mathcal{L}y = 0$, the general solution of the ODE is:
  $$y(x) = Ay_1(x) + By_2(x) + y_p \quad , \quad A, B \text{ const}$$
- 2nd order ODEs require two boundary conditions. The general form of a linear BC is:
  $$\alpha_1 y'(a) + \alpha_2 y(a) = \alpha_3$$
  if $\alpha_3 = 0$, BC is homogeneous
- We can have each complementary function satisfy one BC. By linearity, the superposition will satisfy both
- The Wronskian of two solutions of a 2nd order ODE is a function given by the determinant of the Wronskian matrix:
  $$\boxed{W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$
  ↳ $y_1(x), y_2(x)$ are linearly independent iff $W \neq 0$.
- The Wronskian is an intrinsic property of the ODE and can be calculated before we know $y_1(x), y_2(x)$:
  $$W' = y_1 y_2'' - y_2 y_1'' \quad \leftarrow \text{we know } y_2'' \text{ satisfies the homogeneous ODE.}$$
  $$= y_1(-py_2' - qy_2) - y_2(-py_1' - qy_1)$$
  $$\therefore \quad = -pW \quad \Rightarrow \quad \boxed{W = \exp\left[-\int p(x)\,dx\right]}$$
- Hence if we know only one complementary function, we can find another by first calculating W
  $$y_1 y_2' - y_2 y_1' = W \quad \Rightarrow \quad y_2(x) = y_1(x) \int \frac{W(x)}{[y_1(x)]^2}\,dx$$

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Impulses and Green's functions

- An impulse is defined by $dp = \int_0^{\delta t} F(t)\,dt$. For an instantaneous impulse, we need finite dp as $\delta t \to 0$, which requires $F \to \infty$.
- The Heaviside unit step function is $H(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$
- The delta function can be defined as $\delta(x) = \frac{d}{dx} H(x)$

↳ defining property is $\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)}$

↳ hence $\delta(x)$ can be defined as the limit of certain functions, e.g gaussians as std $\to 0$.

- Derivatives of the delta function can be found via integration by parts:

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x-a)\,dx = \Big[f(x)\,\delta(x-a)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x-a)\,dx$$

$$\Rightarrow \delta^{(k)}[f] = (-1)^k f^{(k)}(0).$$

Greens functions

- If the ODE's forcing function is discontinuous, we solve on either side of the boundary then match

e.g $y'' + y = \delta(x)$ $\therefore$ $y = \begin{cases} A\cos x + B\sin x & x<0 \\ C\cos x + D\sin x & x>0 \end{cases}$

↳ we can integrate both sides of the ODE, assuming y is continuous and bounded.

$$\int_{-\varepsilon}^{\varepsilon} y''\,dx + \int_{-\varepsilon}^{\varepsilon} y\,dx = \int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx$$

↳ Let $\varepsilon \to 0$ $\therefore$ $\int_{-\varepsilon}^{\varepsilon} y\,dx = 0$, hence the matching condition is a jump on the derivative:

$$\left[\frac{dy}{dx}\right]_{x=0^+} = 1 + \left[\frac{dy}{dx}\right]_{x=0^-}$$

- Any forcing function can be treated as an infinite number of spikes (delta functions). So if we know how a system responds to a δ impulse at point ξ, we can convolve this response with the full forcing function to solve the ODE.

↳ Green's function for a specific ODE characterises the response to $\delta(x-\xi)$:

$$\boxed{\begin{array}{l} G(x,\xi) \text{ such that } \mathcal{L}\,G = \delta(x-\xi) \\ \Rightarrow y(x) = \int_0^{\infty} G(x,\xi)\,f(\xi)\,d\xi \end{array}}$$

↳ $G(x,\xi)$ defined for $x \geq 0$, $\xi \geq 0$

↳ G must satisfy the same BCs.

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Series solutions to ODEs

- Consider a homogeneous linear second order ODE:

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0$$

  ↳ $x = x_0$ is an ordinary point is $p(x)$ and $q(x)$ are both analytic at $x = x_0$

  ↳ otherwise $x = x_0$ is a singular point

- A singular point is regular if $(x-x_0)p(x)$ and $(x-x_0)^2 q(x)$ are both analytic at $x = x_0$, else the singular point is irregular.

Series solutions about an ordinary point

- If $x = x_0$ is ordinary, the ODE has two linearly independent power series solutions $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$,

  within the radius of convergence.

- We then have:

$$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$$y' = \sum_{n=0}^{\infty} n a_n (x-x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-x_0)^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x-x_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-x_0)^n$$

convenient to have same power as before

- Since both $p(x)$ and $q(x)$ are analytic, we can write

$$p(x) = \sum_{n=0}^{\infty} p_n (x-x_0)^n \qquad q(x) = \sum_{n=0}^{\infty} q_n (x-x_0)^n$$

↳ power series can be multiplied via

$$\sum_{l=0}^{\infty} A_l (x-x_0)^l \sum_{m=0}^{\infty} B_m (x-x_0)^m = \sum_{n=0}^{\infty} \left[\sum_{r=0}^{n} A_{n-r} B_r \right] (x-x_0)^n$$

↳ hence we can write down a recurrence relation for a_{n+2}, though in practice it may be easier to substitute the power series into a nonstandard form and compare coefficients

- Legendre's equation is:

$$(1-x^2)y'' - 2xy' + l(l+1)y = 0$$

  ↳ $x = 0$ is ordinary so substitute $y = \sum_{n=0}^{\infty} a_n x^n$

$$\therefore \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} \{ -n(n-1) - 2n + l(l+1) \} a_n x^n = 0$$

$$\therefore (n+2)(n+1) a_{n+2} + [-n(n+1) + l(l+1)] a_n = 0$$

$$\Rightarrow a_{n+2} = \frac{(n-l)(n+l+1)}{(n+1)(n+2)} a_n$$

  ↳ the even solution corresponds to $a_0 = 1$, $a_1 = 0$ while the odd solution is obtained by $a_0 = 0$, $a_1 = 1$.

  ↳ these solutions are Legendre polynomials, $P_l(x)$

  ↳ the radius of convergence $= 1$.

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Series solutions about a regular singular point

- If $x = x_0$ is a regular singular point, Fuch's theorem guarantees a solution of the form:

$$\boxed{y = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\sigma}, \quad \sigma \in \mathbb{C}, \quad a_0 \neq 0}$$

↳ this is a Taylor series iff σ is a non-negative integer

↳ there may be either one or two solutions.

- By definition of regularity, we can write

$$(x-x_0)\,p(x) = \sum_{n=0}^{\infty} P_n (x-x_0)^n; \quad (x-x_0)^2 q(x) = \sum_{n=0}^{\infty} Q_n (x-x_0)^n$$

- Near $x = x_0$ we can thus approximate p and q as

$$p = P_0/x-x_0, \quad q = Q_0/(x-x_0)^2$$

$$\therefore \; y'' + \frac{P_0 y'}{x-x_0} + \frac{Q_0 y}{(x-x_0)^2} \approx 0$$

↳ this ODE can be solved by $y = (x-x_0)^\sigma$, where σ satisfies the indicial equation

$$\boxed{\sigma(\sigma-1) + P_0\sigma + Q_0 = 0}$$

- The indicial equation has two (complex) roots

↳ if the roots are equal, the solutions are $(x-x_0)^\sigma$ and $(x-x_0)^\sigma \ln(x-x_0)$

- As with ordinary points, we may not need to formally calculate P_0 and Q_0, we can use Frobenius' method and directly substitute.

↳ e.g Bessel's equation has a regular singular point at $x=0$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0.$$

↳ Let $y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$

$$\Rightarrow \sum_{n=0}^{\infty} \left[(n+\sigma)(n+\sigma-1) + (n+\sigma) - \nu^2\right] a_n x^{n+\sigma} + \sum_{n=0}^{\infty} a_n x^{n+\sigma+2} = 0$$

↳ Then compare coefficients of $x^{n+\sigma}$

$n=0$: $[\sigma^2 - \nu^2]\,a_0 = 0$ ← indicial equation since $a_0 \neq 0$

$n=1$: $[(1+\sigma)^2 - \nu^2]\,a_1 = 0$ ← $a_1 = 0$

$n \geq 2$: $[(n+\sigma)^2 - \nu^2]\,a_n + a_{n-2} = 0$ ← gives recurrence

- If σ_1 and σ_2 differ by an integer, the recurrence may fail for the smaller of the two.

$$y_1 = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\sigma_1} \qquad Re(\sigma_1) \geq Re(\sigma_2)$$

$$y_2 = \sum_{n=0}^{\infty} b_n (x-x_0)^{n+\sigma_2} + c\,y_1 \ln(x-x_0)$$

↳ it is common in science for only one solution to be analytic and the other singular.

↳ it may be easier to use the Wronskian to construct y_2

Sturm-Liouville Theory

- Differential operators are analogous to linear operators.
- The inner product of two piecewise-continuous functions with respect to some weight function $w(x) > 0$

$$\langle u | v \rangle_w = \int_\alpha^\beta u^*(x)\, v(x)\, w(x)\, dx$$

- A differential operator $\tilde{\mathcal{L}}$ is self-adjoint if

$$\langle u | \tilde{\mathcal{L}} v \rangle = \langle \tilde{\mathcal{L}} u | v \rangle$$ ← analogous to Hermitian matrices.

↳ depends on the weight function

↳ generally, the adjoint is found with integration by parts.

$$\left.\langle u | \tilde{\mathcal{L}} v \rangle = \int_\alpha^\beta u^*(x)\, \tilde{\mathcal{L}} v(x)\, dx \right\}$$ IBP transfers derivative

$$= \int_\alpha^\beta [\tilde{\mathcal{L}}^\dagger u(x)]^* v(x)\, dx + \text{boundary terms}$$

- A 2nd order linear differential operator $\mathcal{L}$ is Sturm-Liouville type if:

$$\boxed{\mathcal{L} = -\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x)}$$ ← $\mathcal{L}$ is real

↳ p and q are real functions defined for $\alpha \le x \le \beta$ with $p(x) > 0$ for $\alpha < x < \beta$.

- For suitable B.Cs, the Sturm-Liouville operator is self-adjoint

↳ can be shown by expanding $\langle u | \mathcal{L} v \rangle$ then using IBP

$$\therefore \left[p(x)\left(v\frac{du^*}{dx} - u^*\frac{dv}{dx} \right) \right]_\alpha^\beta = 0$$

↳ functions u, v on which $\mathcal{L}$ operates must satisfy homogeneous B.Cs at $x = \alpha$, $x = \beta$.

- If $\tilde{\mathcal{L}}$ is not of Sturm-Liouville type, there exists a weight function such that $w(x)\tilde{\mathcal{L}} = \mathcal{L}$

↳ let $\tilde{\mathcal{L}} = -P(x)\frac{d^2}{dx^2} - R(x)\frac{d}{dx} - Q(x)$

$$\Rightarrow \mathcal{L} = -wP\frac{d^2}{dx^2} - wR\frac{d}{dx} - wQ$$

↳ consider $\frac{d}{dx}\left(wP\frac{d}{dx}\right) = \frac{d}{dx}(wP) + wP\frac{d^2}{dx^2}$

↳ hence $\mathcal{L}$ is Sturm-Liouville type if

$$P\frac{dw}{dx} + \left(\frac{dP}{dx} - R\right)w = 0$$ ← solve for $w(x)$

↳ i.e. $\tilde{\mathcal{L}}$ is self-adjoint w.r.t $w(x)$, or equivalently $\mathcal{L} = w\tilde{\mathcal{L}}$ is self-adjoint w.r.t the identity weight func.

Eigenfunctions

- An eigenfunction $y(x)$ of an operator $\tilde{\mathcal{L}}$ satisfies $\tilde{\mathcal{L}} y = \lambda y$ where λ is the complex eigenvalue.
- Generally, the eigenvalue equation only has solutions for a discrete (but infinite) set of eigenvalues λ_n, $n \in \mathbb{Z}^+$.

e.g. $\mathcal{L} = -\frac{d^2}{dx^2}$ is Sturm-Liouville type with $p(x) = 1$, $q(x) = 0$

↳ Eigenvalue equation: $y'' + \lambda y = 0$.

↳ if $y(0) = y(\pi) = 0$, $y_n(x) = B\sin nx$ with $\lambda = n^2, n \in \mathbb{Z}^+$

↳ by convention we normalise the resulting eigenfunctions.

- It can be shown that a self-adjoint operator has real eigenvalues

$$\lambda\langle y|y\rangle = \langle y|\lambda y\rangle = \langle y|\mathcal{L}y\rangle = \langle \mathcal{L}y|y\rangle = \langle \lambda y|y\rangle$$
$$\Rightarrow \lambda\langle y|y\rangle = \lambda^*\langle y|y\rangle \Rightarrow \lambda = \lambda^*$$

- The eigenfunctions of a self-adjoint operator corresponding to distinct eigenvalues are orthogonal.
  - ↳ proof same as analogous claim for Hermitian matrices
  - ↳ suppose y_1, y_2 are eigenfunctions with distinct eigenvalues λ_1, λ_2
    $$\lambda_2\langle y_1|y_2\rangle = \lambda_1\langle y_1|y_2\rangle \Rightarrow \langle y_1|y_2\rangle = 0.$$
  - ↳ even for repeated eigenvalues, an orthonormal set of eigenfunctions can always be constructed.
- Legendre's equation can be written as a Sturm-Liouville eigenvalue equation: $\mathcal{L} = -\frac{d}{dx}\left[(1-x^2)\frac{d}{dx}\right]$, $\lambda = l(l+1)$
  - ↳ the only finite nonzero solutions at $x = \pm 1$ are the terminating Legendre polynomials $P_l(x)$
  - ↳ $\{P_l(x)\}$ is orthogonal, but not orthonormal with $P_l(1) = 1$
- The eigenfunctions of a self-adjoint operator are complete – they can be used as basis functions in an infinite series to repr any $f(x)$ that satisfies the B.Cs.
  $$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$
  - ↳ the coefficients are found by exploiting orthogonality.
    $$a_n = \langle y_n|f\rangle_w$$
  $$\therefore f(x) = \int_\alpha^\beta f(\xi)\left[w(\xi)\sum_{n=1}^{\infty} y_n(x) y_n^*(\xi)\right]d\xi$$

completeness relation

$$\Rightarrow \boxed{w(\xi)\sum_{n=1}^{\infty} y_n(x) y_n^*(\xi) = \delta(x-\xi)}$$

completeness relation

can swap x, ξ

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Solving ODEs with eigenfunction expansions

- Consider $\mathcal{L}y = f(x)$ with $\mathcal{L}$ in Sturm-Liouville form.
- The completeness relation can be used to construct a Green's function (for nonzero λ_n)
  $$\boxed{G(x,\xi) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} y_n(x) y_n^*(\xi)}$$
  - ↳ $\mathcal{L}_x G(x,\xi) = \sum_{n=1}^{\infty} y_n^*(\xi)\cdot\frac{1}{\lambda_n}\mathcal{L}y_n(x)$
    $$= \sum_{n=1}^{\infty} y_n^*(\xi)\, w(x)\, y_n(x) = \delta(x-\xi)$$
  - ↳ note that $G(x,\xi) = G^*(\xi,x)$
- If there is a solution to $\mathcal{L}y = 0$ satisfying the B.Cs, then any nonzero force results in infinite response
  - ↳ this is resonance; equivalent to having $\lambda_n = 0$
  - ↳ if one eigenvalue is much smaller than the others, the result will be near-resonant:
    $$y(x) = \int_\alpha^\beta \sum_{n=1}^{\infty}\frac{1}{\lambda_n} y_n(x) y_n^*(\xi) f(\xi)\, d\xi \simeq \frac{y_1(x)}{\lambda_1}\langle y_1|f\rangle$$

- Rather than using the Green's function, we may be able to construct the solution directly.
  $$y = \sum b_n y_n, \quad f = \sum a_n y_n$$
  $$\therefore \mathcal{L}y = f \Rightarrow \sum b_n \lambda_n y_n = \sum a_n y_n$$

Approximation with eigenfunction expansions

- We may wish to approximate a solution as a finite linear combination of eigenfunctions $f(x) \approx \sum_{n=1}^{N} a_n y_n(x)$
- Coefficients should minimise the total error:

$$S(a_1, a_2, \dots, a_N) = \left\| f(x) - \sum_{n=1}^{N} a_n y_n(x) \right\|_w^2$$

↳ by expanding the norm and taking partials $\partial S / \partial a_k$, it can be shown that S is minimised for $a_k = \langle y_k | f \rangle_w$

↳ i.e same as infinite case

$$\Rightarrow S_n = \|f\|_w^2 - \sum_{n=1}^{N} |a_n|^2$$

↳ $S_n \geq 0$, from which we have Bessel's inequality:

$$\|f\|_w^2 \geq \sum_{n=1}^{N} |a_n|^2$$

↳ in the limit this becomes equality, generalising Parseval's thm:

$$\boxed{\|f\|_w^2 = \sum_{n=1}^{\infty} |a_n|^2}$$