Partial Differential Equations

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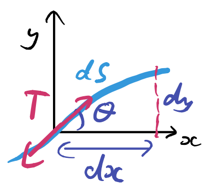
- A PDE is any equation of the form
  $F(u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \dots) = 0$
  - ↳ linear if F depends linearly on u terms, in which case we write $\mathcal{L}u = f$.
  - ↳ obeys similar principles to linear ODEs, i.e the general solution can be constructed from particular + complementary.

- The diffusion equation:
  - ↳ for a conserved quantity with concentration Q and flux density $\underline{F}$, conservation implies $\frac{\partial Q}{\partial t} + \nabla \cdot \underline{F} = 0$
  - ↳ but the flux depends on the conc. gradient: $\underline{F} = -\lambda \nabla Q$ Fick's law

  $$\Rightarrow \boxed{\frac{\partial Q}{\partial t} = \lambda \nabla^2 Q}$$ ← reduces to Laplace's equation in the steady state
  - ↳ this applies to heat conduction, with $Q = CT$.

- The wave equation:
  - ↳ $m\ddot{y} = F_y \Rightarrow \rho ds \frac{\partial^2 y}{\partial t^2} = \frac{\partial F_y}{\partial x} dx$
  - ↳ $F_y \approx T \frac{\partial y}{\partial x}$ and $\delta s \approx \delta x$

  $$\Rightarrow \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad c = \sqrt{\frac{T}{\rho}}$$



- Types of boundary condition:
  - ↳ Dirichlet: u specified
  - ↳ Neumann: $\partial u / \partial x$ specified
  - ↳ Mixed: some LC of u and $\frac{\partial u}{\partial x}$ specified

- Separation of variables seeks solutions of the form
  $u(x,t) = X(x)T(t)$
  - ↳ substitute in and rearrange so LHS only contains T, t, RHS only contains X, x.
  - ↳ each side must then equal a constant, so we have 2 ODEs.
  - ↳ for each value of this constant, there may be a different solution. In general, we must sum all.

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Laplace's and Poisson's Equations

- Poisson's equation is a 2nd order PDE: $\boxed{\nabla^2\psi = \rho(\underline{x})}$
- For the special case of $\rho = 0$, it reduces to Laplace's equation.
- Physical examples:
  - ↳ steady state diffusion
  - ↳ electrostatics: $\nabla^2\phi = -\rho/\varepsilon_0$
  - ↳ gravitation: $\nabla^2\phi = 4\pi G\rho$
  - ↳ ideal irrotational fluid flow
- The general solution to Laplace's equation can be written as a LC of a set of basis solutions.

Polar coordinates

- Laplace's equation: $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} = 0$
- For separable solutions: $\frac{r}{R}\frac{d}{dr}(rR') = -\frac{1}{\Phi}\Phi'' = \lambda$

- If ψ is a physical quantity, it must be 2π-periodic in ϕ. Else if it is a potential, ψ' must be periodic.
  - ↳ so $\lambda = n^2$, and we have cases $n=0$, $n\neq 0$.

$$\boxed{\begin{aligned} \Phi(\phi) &= \begin{cases} A + B\phi & n=0 \\ A\cos n\phi + B\sin n\phi & n\neq 0 \end{cases} \\ \Rightarrow R(r) &= \begin{cases} C\ln r + D, & n=0 \\ Cr^n + Dr^{-n}, & n\neq 0 \end{cases} \end{aligned}}$$

R can be solved using $r=e^t$ sub.

- The general solution is then:

$$\psi(r,\phi) = \underbrace{(A_0 + B_0\phi)}_{\text{normally disappears due to periodicity.}}(C_0\ln r + D_0) + \sum_{n=1}^{\infty}(A_n r^n + C_n r^{-n})\cos n\phi + \sum_{n=1}^{\infty}(B_n r^n + D_n r^{-n})\sin n\phi$$

Spherical coordinates (axisymmetric)

- Axisymmetry implies independence of ϕ (i.e. surface of revolution around z-axis)
- Laplace's equation: $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) = 0$
- Separate: $\frac{1}{R}\frac{d}{dr}(r^2R') = -\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}(\sin\theta\,\Theta') = \lambda$
- With the substitution $u=\cos\theta$, $\frac{d}{d\theta} = -\sin\theta\frac{d}{du}$

$$\Rightarrow \frac{d}{du}\left((1-u^2)\frac{d\Theta}{du}\right) + \lambda\Theta = 0$$

  - ↳ i.e. the angular part satisfies Legendre's equation.
  - ↳ $\lambda = l(l+1)$, $l = 0,1,2$ and $\boxed{\Theta_l = P_l(\cos\theta)}$
- The general solution is then:

$$\boxed{\psi(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)}$$

- Fitting B.Cs may require integrating over the Legendre polynomials exploiting orthogonality: $\int_{-1}^{1}P_m(x)P_n(x)\,dx = \frac{2}{2m+1}\delta_{mn}$.

Uniqueness

- Suppose there are two solutions ϕ_1, ϕ_2. Consider the difference of these solutions $\psi = \phi_1 - \phi_2$

$$\nabla\cdot(\psi\nabla\psi) = (\nabla\psi)\cdot(\nabla\psi) + \psi\nabla^2\psi$$
$$= |\nabla\psi|^2$$

{zero because Poisson's equation is linear

↳ applying the divergence theorem,
$$\int_V |\nabla\psi|^2 dV = \oint_S (\psi\nabla\psi)\cdot d\underline{S}$$

↳ given Dirichlet boundary conditions, i.e $\phi = f(\underline{r})$ on S,
$\psi = 0$ when evaluated on S $\Rightarrow \int_V |\nabla\psi|^2 dV = 0$

↳ $\psi = 0$ on S and $\nabla\psi = 0$ inside $\Rightarrow \psi = 0$

↳ hence $\phi_1 = \phi_2$ so any solution is unique.

- For Neumann boundary conditions, $\hat{\underline{n}}\cdot\nabla\phi = 0$ on S; the proof is similar.

The Fundamental solution

- Consider Poisson's equation with Dirichlet conditions on a surface S which bounds a volume V.
- The Green's function for this problem is given by:

$$\boxed{\begin{aligned} \nabla^2 G(\underline{r},\underline{r}') &= \delta^{(3)}(\underline{r}-\underline{r}'), \quad \underline{r}\in V \\ G(\underline{r},\underline{r}') &= 0, \quad \underline{r}\notin V \end{aligned}}$$

↳ $\delta^{(3)}(\underline{r}-\underline{r}') \equiv \delta(x-x')\delta(y-y')\delta(z-z')$

↳ $$\int_V f(\underline{r})\delta^{(3)}(\underline{r}-\underline{r}')\,dV = \begin{cases} f(\underline{r}') & r'\in V \\ 0 & r'\notin V \end{cases}$$

↳ the Green's function is symmetric, i.e $G(\underline{r},\underline{r}') = G(\underline{r}',\underline{r})$

↳ if V is all space, G is called the *fundamental solution*.

- The Green's function is the potential due to a point charge at $\underline{r}'$, hence the symmetry is obvious.
- For Neumann B.Cs, G needs a different form on S

$$\boxed{\frac{\partial G}{\partial n} = \frac{1}{A}, \quad A \equiv \oint_S dS}$$

← before we had $G = 0$ on S

↳ this arises because $\int_S \frac{\partial G}{\partial n} dS = \int_S \nabla G\cdot\hat{\underline{n}}\,dS = \int_V \nabla^2 G\,dV$, but $\int_V \nabla^2 G\,dV = \int_V \delta^{(3)}(\underline{r}-\underline{r}')\,dV = 1$.

- Consider a point charge at the origin of 3D space. The fundamental solution is given by $\nabla^2 G = \delta^{(3)}(\underline{r})$ with $G \to 0$ as $|\underline{r}| \to \infty$.
- By symmetry, G is radial: $\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = 0$ ← for $r \neq 0$

$$\Rightarrow G(r) = A + \frac{C}{r}$$

↳ $A = 0$ to satisfy B.C at ∞.

↳ C can be found by integrating $\nabla^2 G$ over a small sphere

$$\underbrace{\int_{r<\varepsilon} \nabla^2 G\,dV}_{=1} = \oint_{r=\varepsilon} \frac{\partial G}{\partial r} dS = -\frac{C}{\varepsilon^2}\oint_{r=\varepsilon} dS = -4\pi C$$

$$\Rightarrow C = -\frac{1}{4\pi}$$

- Hence if we shift the origin to $\underline{r}'$: $\boxed{G(\underline{r},\underline{r}') = -\frac{1}{4\pi|\underline{r}-\underline{r}'|}}$

- For a line of charge, the problem is equivalent to finding a 2D Green's Function, $\nabla^2 G = \delta^{(2)}(\underline{r})$
$$\Rightarrow G(r) = A + C\ln r$$
  - ↳ we can no longer force $G \to 0$ as $r \to \infty$
  - ↳ to find C, we use the 2D divergence theorem for a small circle of radius ε.
$$\int_{r<\varepsilon} \nabla^2 G \, dA = 2\pi C \quad \Rightarrow \quad C = 1/2\pi$$
  - ↳ hence G is only defined to within an additive constant:
$$G = \frac{1}{2\pi} \ln|\underline{r} - \underline{r}'| + \text{const}.$$

The method of images

- The method of images can be used to find G in some other simple geometries (the fundamental solution only applies in all-space).
- 3D half-space, i.e. $z > 0$:
  - ↳ need G to satisfy $\nabla^2 G = \delta^{(3)}(\underline{r} - \underline{r}')$, $\underline{r} \in D$ ← domain
$$G \to 0 \text{ as } |r| \to \infty$$
$$G = 0 \text{ at } z = 0 \quad \leftarrow \text{new B.C (Dirichlet)}$$
  - ↳ we can construct a solution in $\mathbb{R}^3$ that fits this by placing an image charge at $\underline{r}'' = (x', y', -z')$, with opposite sign $\therefore \nabla^2 G = \delta^{(3)}(\underline{r} - \underline{r}') + \delta^{(3)}(\underline{r} - \underline{r}'')$
$$\Rightarrow G(\underline{r}, \underline{r}') = -\frac{1}{4\pi|\underline{r} - \underline{r}'|} + \frac{1}{4\pi|\underline{r} - \underline{r}''|}$$
  - ↳ by uniqueness, this is *the* solution
  - ↳ if we instead had Neumann bounds, i.e. $\left.\frac{\partial G}{\partial z}\right|_{z=0} = 0$, we could use an image charge with the same sign.

- The image for a charge in a sphere (radius a) has strength $-\frac{a}{r'}$ and is located at r'': $\frac{r''}{a} = \frac{a}{r'}$. It can be shown that this gives $G = 0$ when $r = a$ as needed.
- For a circle, the image has strength -1 and is located again at the inverse point $\frac{r''}{a} = \frac{a}{r'}$.

The integral solution of Poisson's equations

- For smooth functions ϕ, ψ in volume V enclosed by S, **Green's identity** states
$$\boxed{\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) \, dV = \oint_S \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) dS}$$
  - ↳ follows directly from the divergence theorem
  - ↳ can be used in 2D, for surface S enclosed by curve C
- This can be used to find the integral solution once we know the Green's function, letting $\psi \equiv G$ and solving $\nabla^2 \phi = \rho(\underline{r})$.
- For Dirichlet B.Cs, $\phi = f$ on S
  - ↳ Green's identity gives
$$\int_V (\phi \, \delta^{(3)}(\underline{r} - \underline{r}') - G\rho) \, dV = \oint f \nabla G \cdot \hat{\underline{n}} \, dS$$

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$$\Rightarrow \boxed{\phi(\underset{\sim}{r}') = \int_V \rho(\underset{\sim}{r})\, G(\underset{\sim}{r},\underset{\sim}{r}')\,dV + \oint_S f(\underset{\sim}{r}) \frac{\partial G}{\partial n}\,dS}$$

↳ if V is all space and $f \to 0$ as $|\underset{\sim}{r}| \to \infty$, then provided G and ϕ decrease fast enough $(\propto \frac{1}{r})$:

$$\phi(\underset{\sim}{r}') = \int_{\mathbb{R}^3} \rho(\underset{\sim}{r})\, G(\underset{\sim}{r},\underset{\sim}{r}')\,dV$$

- For Neumann B.Cs, $\frac{\partial G}{\partial n} = \frac{1}{A}$, $\frac{\partial \phi}{\partial n} = g(\underset{\sim}{r})$ on S: arbitrary

$$\phi(\underset{\sim}{r}') = \int_V \rho(\underset{\sim}{r})\, G(\underset{\sim}{r},\underset{\sim}{r}')\,dV - \oint_S g(\underset{\sim}{r})\, G(\underset{\sim}{r},\underset{\sim}{r}')\,dS + C$$