The Black-Litterman Model

Equilibrium

- Generally, equilibrium means that supply = demand.
- With the Quadratic Utility function and a risk-free asset, the equilibrium portfolio is the CAPM Market portfolio.
- CAPM assumes:
  - every investor agrees on $\mu$ and $\Sigma$ and maximises utility
  - unique risk-free rate of borrowing and lending
  - normally distributed returns
- then $E(r_i) = r_f + \beta_i(E(r_m) - r_f)$
- All investors should thus hold the market portfolio

Prior returns

- BL computes the mkt-implied returns by reverse optimisation
  \[ U = w^T \Pi = \frac{1}{2} w^T \Sigma w \]
  \[ \text{w} \] without constraints, this is easy to solve: \[ \nabla U = 0 \Rightarrow \Pi = \Sigma \Sigma \]
  \[ \text{w} \] these returns are likely to be 'healthier' than mean historical.
  \[ \text{w} \] can be estimated from the CAPM: \[ \Sigma = \frac{E(r) - \mu}{\sigma^2} \]
- The cov matrix of expected returns $\Sigma_{nn}$ is modeled by $\tau \Sigma$,
  where $\tau$ is some small scalar (unc in mean << unc in returns)
The BL prior is then: \( \mathbb{E}(r) \sim \mathcal{N}(\Pi, \Sigma) \), with future returns generated by \( r \sim \mathcal{N}(\mathbb{E}(r), \Sigma) \).

**Investor's views**

BL allows for \( K \) views on \( N \) assets, where each view can either be absolute or relative. From these views, we must construct three matrices:

\( Q \in \mathbb{R}^{K \times 1} \) is the vector of views,

\( P \in \mathbb{R}^{K \times N} \) are the asset weights for each view (sum to 0 if relative, 1 otherwise) — a.k.a picking matrix,

\( \Omega \in \mathbb{R}^{N \times N} \) is the diagonal matrix of view variances,

\( \Sigma^{-1} \) is the investor's confidence.

i.e. \( \mathbb{E}E(r) = Q + \varepsilon, \varepsilon \sim \mathcal{N}(0, \Omega) \).

**Specifying \( \Omega \)**

- He and Litterman (1992) suggest \( \Omega = \text{diag} ( P(\pi \Sigma) P^T ) \), i.e. view variance α variance of asset returns.
- Alternatively, if a confidence interval is specified, we can extract a variance (assuming normal data).
- Idzorek's method lets investors specify views with a % confidence

\( \Omega = \alpha P \Sigma P^T \), \( \alpha = \frac{1 - \text{conf}}{2 \text{conf}} \),
The BL-formula from Bayes' Theorem

\[ \Pr(A \mid B) = \frac{\Pr(B \mid A) \Pr(A)}{\Pr(B)} \]

- posterior dist.

In the case of BL:
\[ \Pr(\text{EC}(r) \mid \text{PEC}(r)) = \frac{\Pr(\text{PEC}(r) \mid \text{EC}(r)) \Pr(\text{EC}(r))}{\Pr(\text{PEC}(r))} \]

- updated exp. returns

\[ \Rightarrow \text{but all of these are normal dists, i.e.:} \]
\[ \text{EC}(r) \mid \text{PEC}(r) \sim \mathcal{N}(\mu^*, \Sigma) \]

\[ \Rightarrow \text{the goal of the BL formula is to compute } \mu^* \]

- \( \text{EC}(r) \sim \mathcal{N}(\pi, \Sigma) \) and \( \text{PEC}(r) \mid \text{EC}(r) \sim \mathcal{N}(Q, \Omega) \)

- Then we can write down the pdfs, e.g.
\[ f(\text{EC}(r)) = \frac{1}{\sqrt{(2\pi)^n |\Sigma|}} \exp \left[ -\frac{1}{2} (\text{EC}(r) - \pi)^T \Sigma^{-1} (\text{EC}(r) - \pi) \right] \]

- These can be substituted directly into Bayes formula. Expanding inside the exponent (dropping the \(-\frac{1}{2}\))
\[ (\text{EC}(r) - \pi)^T \Sigma^{-1} (\text{EC}(r) - \pi) + (\text{PEC}(r) - Q)^T \Omega^{-1} (\text{PEC}(r) - Q) \]
\[ = \text{EC}(r)^T (\Sigma)^{-1} \text{EC}(r) - \text{EC}(r)^T (\Sigma)^{-1} \pi - \pi^T (\Sigma)^{-1} \text{EC}(r) + \pi^T (\Sigma)^{-1} \pi + \text{EC}(r)^T \Omega^{-1} \text{PEC}(r) - \text{EC}(r)^T \Omega^{-1} Q - Q^T \Omega^{-1} \text{PEC}(r) + Q^T \Omega^{-1} Q \]

- We can then group equal terms (using symmetry of \(\Omega\) and \(\Sigma\)) and factorise \(\text{EC}(r)^T \text{EC}(r)\) and \(\text{EC}(r)\). We introduce symbols \(C, H, A:\)
\[ C = (\tau \Sigma)^{-1} \Pi + \rho^T \Omega^{-1} Q \]
\[ A = Q^T \Omega^{-1} Q + \Pi^T (\tau \Sigma)^{-1} \Pi \]

Then the exponent becomes:
\[ E(r)^T \Sigma E(r) - 2C^T E(r) + A \]
\[ = (HE(r))^T H^{-1} H E(r) - 2C^T H^{-1} H E(r) + A \]
\[ = (HE(r) - C)^T H^{-1} (HE(r) - C) + A - C^T H^{-1} C \]
\[ = (E(r) - H^{-1} C)^T H (E(r) - H^{-1} C) + A - C^T H^{-1} C \]
\[ \Rightarrow \quad \text{posterior mean: } \mu^* = ( (\tau \Sigma)^{-1} + \rho^T \Omega^{-1} \rho )^{-1} ( (\tau \Sigma)^{-1} \Pi + \rho^T \Omega^{-1} Q ) \]
\[ \text{posterior covariance: } M = ( (\tau \Sigma)^{-1} + \rho^T \Omega^{-1} \rho )^{-1} \]

However, this covariance is for the expected returns. The posterior estimate for the return dist is \( \Sigma^* = \Sigma + M \).

The \( \tau \) parameter

\( \tau \) is measures confidence in the prior estimates.

Can be estimated using confidence intervals: pick a value of \( \tau \), compute the 95% or 99% confidence interval and see whether the range of \( E(r) \) is reasonable.

Alternatively, we can set \( \tau \sim \frac{1}{\tau} \), because variance is inversely proportional to the number of samples.