Fluid Equations

- A fluid element must be:
  - ↳ small enough so that macroscopic properties $\sim$ const
  - ↳ large enough to have a large number of particles.
- Collisional fluids have a small mean-free-path λ:
  - ↳ particles maximise entropy
  - ↳ well-defined pressure $p = p(\rho, T)$ ← equation of state
- Collisionless fluids have non-local effects and depend on I.Cs.

- Two frameworks for describing fluids:
  - ↳ Eulerian → consider fluid properties as time-varying fields, e.g $\rho(\underline{r},t)$, $p(\underline{r},t)$, $T(\underline{r},t)$, $\underline{v}(\underline{r},t)$
  - ↳ Lagrangian → perspective of a particular fluid element as time progresses (co-moving frame)
- Consider how quantity $Q(\underline{r},t)$ changes in the Lagrangian picture

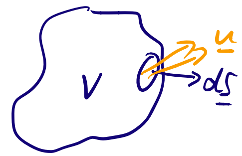
$$\frac{DQ}{Dt} = \lim_{\delta t \to 0}\left[\frac{Q(\underline{r}+\delta\underline{r}, t+\delta t) - Q(\underline{r}, t)}{\delta t}\right]$$

$$\Rightarrow \boxed{\frac{DQ}{Dt} = \frac{\partial Q}{\partial t} + \underbrace{\underline{u}\cdot\nabla Q}_{\text{convective derivative}}}$$ ← Lagrangian ⟷ Eulerian

convective derivative: gradient projected onto flow

- 3 ways to describe particle trajectories:
  - ↳ streamlines show the velocity field at a given time, i.e shows instantaneous tangents
  - ↳ pathlines show the paths taken by individual fluid elements
  - ↳ streaklines connect all points that passed through a particular reference point (e.g after releasing a drop of dye).
  - ↳ all 3 coincide if $\frac{\partial \underline{u}}{\partial t} = 0 \Rightarrow$ steady flow

Conserved quantities



- Conservation of mass:
  - ↳ $\frac{dm}{dt} = -$ rate of outflow

$$\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_S \rho \underline{u}\cdot d\underline{S}$$

div. theorem

$$\Rightarrow \int_V \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u})\right) dV = 0$$

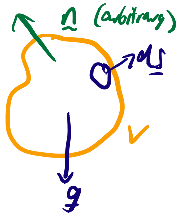
$$\Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0}$$ → continuity equation (Eulerian)

  - ↳ easy to derive Lagrangian form: $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho$

$$\Rightarrow \boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0}$$

- For an incompressible flow, $\frac{D\rho}{Dt} = 0 \Rightarrow \nabla\cdot\underline{u} = 0$, i.e the fluid is divergence-free.

- The forces in a fluid are described with the stress tensor σ_{ij}
  - ↳ $dF_i = \sigma_{ij}\, dS_j$
  - ↳ for an isotropic fluid, $\sigma_{ij} = p\delta_{ij} \Rightarrow dF = p\, d\underline{S}$
- Derive cons momentum by considering a fluid element subject to gravity and internal pressure

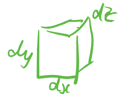


  - ↳ consider all quantities projected onto $\hat{n}$
  - ↳ pressure force $= -\int_S p\,\hat{n}\cdot d\underline{S} = -\int_V \hat{n}\cdot\nabla p\, dV$
  - ↳ NII : $\left(\frac{D}{Dt}\int_V \rho \underline{u}\, dV\right)\cdot\hat{n} = -\int_V \hat{n}\cdot\nabla p\, dV + \int_V \rho \underline{g}\cdot\hat{n}\, dV$
  - ↳ Fluid element so $\int dV \to \delta V$

$$\boxed{\begin{aligned} \rho\frac{D\underline{u}}{Dt} &= -\nabla p + \rho \underline{g} \\ \rho\frac{\partial \underline{u}}{\partial t} + \rho(\underline{u}\cdot\nabla)\underline{u} &= -\nabla p + \rho \underline{g} \end{aligned}}$$

→ momentum equations

→ easier to derive with box



- Consider the Eulerian rate of chg of momentum density

$$\partial_t(\rho u_i) = \rho\partial_t u_i + u_i\partial_t \rho \quad \text{sub cont \& mom. equations}$$
$$= -\rho u_j \partial_j u_i - \partial_j P_{ij} + \rho g_i - u_i\partial_j(\rho u_j)$$
$$= -\partial_j\big(\underbrace{\rho u_i u_j + p\delta_{ij}}_{\equiv\, \sigma_{ij}}\big) + \rho g_i$$

  - ↳ rewrite as $\partial_t(\rho \underline{u}) = -\nabla\cdot\big(\underbrace{\rho\underline{u}\otimes\underline{u} + p\underline{\underline{I}}}_{\text{Flux of momentum density}}\big) + \rho\underline{g}$
  - ↳ $\rho u_i u_j$ is a 'ram' pressure due to momentum flux of the bulk flow

Gravitation

- $\underline{g}(\underline{r}) = -G\int_V \rho(\underline{r}')\, \frac{\underline{r}-\underline{r}'}{|\underline{r}-\underline{r}'|^3}\, dV'$
- Gives Poisson's equation: $\begin{cases} \nabla\cdot\underline{g} = -\nabla^2\Phi = -4\pi G\rho \\ \int_S \underline{g}\cdot d\underline{S} = -4\pi G M_{enc} \end{cases}$
- Potential of a spherically-symmetric system:

$$g(r) = \frac{GM_{enc}}{r^2} = \frac{d\Phi}{dr}$$
$$\Rightarrow \Phi = \int_\infty^{r_0} \frac{G}{r^2}\left(\int_0^r 4\pi\rho(r')r'^2 dr'\right) dr$$
$$\Phi(r) = -\frac{GM(r_0)}{r_0} + \int_\infty^{r_0} 4\pi G\rho(r) r\, dr$$

- The GPE of a system is $\Omega = \frac{1}{2}\int \rho(\underline{r})\,\Phi(\underline{r})\, dV$
- Consider the moment of inertia of a composite system ← arbitrary origin

$$I = \sum_i m_i r_i^2 \Rightarrow \frac{1}{2}\frac{d^2 I}{dt^2} = \sum_i\left(\underline{r}_i\cdot\underline{F}_i + m_i\dot{\underline{r}}_i^2\right)$$

  - ↳ $\sum_i \underline{r}_i\cdot\underline{F}_i$ is the GPE ← assume isolated system and forces are either local or gravitational
  - ↳ $\sum_i m_i \dot{\underline{r}}_i^2$ is twice the KE
  - ↳ for a system in the steady state, $\frac{d^2 I}{dt^2} = 0$
  - ↳ combine to give the Virial theorem : $\boxed{2T + \Omega = 0}$
- The Virial thm relates mass, velocity, size:

$$T = \tfrac{1}{2}M\langle v^2\rangle,\quad \Omega = -\frac{GM^2}{\bar{r}} \Rightarrow \langle v^2\rangle = \frac{GM}{\bar{r}} \leftarrow \text{mass-weighted avg size.}$$

  - ↳ $E_{tot} = T + \Omega = -T = -\frac{1}{2}M\langle v^2\rangle$, so as 'temp' T (higher $\langle v^2\rangle$), energy decreases ⇒ negative heat capacity
  - ↳ this is why structures form from smooth ICs.

Equations of State

- We have 3 scalar + 1 vector unknowns: $\rho, \underline{u}, p, \Psi$, but only 3 eqs (continuity, momentum, Poisson).
- The equation of state provides the additional constraint.
- For a barotropic fluid, pressure is only a function of density: $p = p(\rho)$
  - ↳ e.g electron degeneracy pressure: $p \propto \rho^{5/3}$
  - ↳ e.g isothermal ideal gas: $p \propto \rho$. This occurs when strong heating and strong cooling balance at a well-defined temp.
  - ↳ e.g adiabatic ideal gas $p = K\rho^{\gamma}$.
- Fluid elements may each be adiabatic ($p = K\rho^{\gamma}$ with K const), but K may vary between elements → isentropic if all have same K (because $\ln K \propto S/m$).

The Energy Equation

- From the 1st law of thermo, $\frac{DE}{Dt} = \frac{dQ}{dt} + \frac{DW}{Dt}$
  - ↳ $dW = -p\,dV \Rightarrow \frac{DW}{Dt} = -p\frac{D}{Dt}\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\frac{D\rho}{Dt}$ ← for unit mass, $V = \frac{1}{\rho}$
  - ↳ $\frac{dQ}{dt} \equiv -\dot{Q}_{cool} \Rightarrow \frac{DE}{Dt} = \frac{p}{\rho^2}\frac{D\rho}{Dt} - \dot{Q}_{cool}$
- The total energy of a fluid is $\boxed{E = \rho(\underbrace{\tfrac{1}{2}u^2}_{KE} + \underbrace{\Psi}_{PE} + \underbrace{\varepsilon}_{\text{Internal}})}$

$$\Rightarrow \frac{DE}{Dt} = \frac{D\rho}{Dt}\frac{E}{\rho} + \rho\left(\underline{u}\cdot\nabla\underline{u} + \frac{D\Psi}{Dt} + \frac{p}{\rho^2}\frac{D\rho}{Dt} - \dot{Q}_{cool}\right)$$

↳ write everything in Eulerian, use known equations

$$\boxed{\frac{\partial E}{\partial t} + \nabla\cdot[(E+p)\underline{u}] = \rho\frac{\partial\Psi}{\partial t} - \rho\dot{Q}_{cool}}$$ ← Energy equation

↳ LHS describes change in total energy due to the divergence of the enthalpy flux $(E+p)$.

↳ RHS contains sources of energy. If no external sources, $\frac{\partial E}{\partial t} = \text{div}(\text{enthalpy flux})$.

Heating and cooling ← Astrophysical examples

- Most cooling processes involve radiation.
  1. Collisionally excited atomic line radiation: ← electron-ion collision
     - ↳ collision excites atom, which later emits a photon with energy χ
     - ↳ luminosity per unit volume: $L_c \propto \underbrace{n_e n_{ion}}_{\text{num. collisions}} \underbrace{e^{-\chi/k_B T}\chi/\sqrt{T}}_{\text{energy per collision.}}$
     - ↳ divide by ρ to get $\dot{Q}$ (which is per unit mass)
  2. Recombination emission:
     - ↳ Free electron captured in high energy state
     - ↳ cascades down, releasing photons in the process.
     - ↳ as above, $\dot{Q} \propto \rho f(T)$
  3. Free-free emission (bremsstrahlung): electrons accelerated by nuclei so radiate. $\dot{Q} = \Lambda_0 \rho\sqrt{T}$

- Heating can occur internally, e.g shocks or viscous flows.
- Cosmic rays are an external source of heat, with $\dot{Q} \propto$ ray flux
- Combine heating and cooling to get $\boxed{\dot{Q}_{cool} = A\rho T^{\alpha} - H}$ ← CR cooling

Energy transport

diffusion process

- Thermal conduction: transfer of thermal energy down temp. gradients
  - ↳ energy flux: $\underline{F}_{cond} = -K\nabla T$
  - ↳ so the rate of change of energy density is $-\nabla\cdot\underline{F}_{cond} = \kappa\nabla^2 T$
  - ↳ important in white dwarfs, supernova shocks
- Radiation transport:
  - ↳ in optically thick systems, radiation transports energy rather than cooling
  - ↳ radiative diffusion: $\underline{F}_{rad} \propto -\nabla\mathcal{E}_{rad}$
- Convection → circulating fluid motion (important in stars)

Stellar Fluids

- In static equilibrium, $\underline{u}=0$, $\frac{\partial}{\partial t}=0$
  - ↳ momentum equation ⇒ $\boxed{\nabla p = -\rho\nabla\Psi}$ ← HSE
  - ↳ i.e. pressure forces must balance gravity
- For a self-gravitating system, HSE must be solved together with Poisson's equation.

- Example: isothermal atmosphere with constant external $\underline{g}$
  - ↳ $\underline{g} = -g\hat{\underline{z}}$, $p = \frac{R^*T}{\mu}\rho = A\rho$

    $\Rightarrow A\nabla\rho = -\rho g\hat{\underline{z}} \Rightarrow \frac{A\,d\rho}{\rho} = -g$
  - ↳ $\ln\rho = -\frac{gz}{A} + \text{const} \Rightarrow \rho = \rho_0 e^{-\frac{g}{A}z}$
  - ↳ good model for Earth atmosphere
- Example: isothermal self-gravitating slab
  - ↳ $A\nabla\rho = -\rho\nabla\Psi \Rightarrow A\frac{d\rho}{dz} = -\rho\frac{d\Psi}{dz} \Rightarrow \Psi = -A\ln(\rho/\rho_0) + \Psi_0$

    $\Rightarrow \rho = \rho_0 e^{-(\Psi-\Psi_0)/A}$
  - ↳ to proceed, use Poisson: $\frac{d^2\Psi}{dz^2} = 4\pi G\rho_0 e^{-(\Psi-\Psi_0)/A}$
  - ↳ can model galaxy

- Model a star as a spherically-symmetric self-gravitating system in HSE.
  - ↳ HSE in spherical polars: $\frac{dp}{dr} = -\rho\frac{d\Psi}{dr}$
  - ↳ $\rho > 0$ so p is a monotonically decreasing function of Ψ
  - ↳ $\frac{dp}{dr} = \frac{dp}{d\Psi}\frac{d\Psi}{dr} = -\rho\frac{d\Psi}{dr} \Rightarrow \rho = -\frac{dp}{d\Psi}$ ← also monotonic

↳ $p=p(\Psi)$, $\rho=\rho(\Psi)$ $\Rightarrow$ $p=p(\rho)$ $\Rightarrow$ stars are barotropic

- A useful family of barotropes is $p = K\rho^{1+1/n}$, where $n=n(\rho)$
  - ↳ polytropes have $n=\text{const}$
  - ↳ if the star is isentropic (e.g. fully convective), $1+\frac{1}{n}=\gamma\equiv\frac{C_v}{C_p}$, so the polytrope equation is the adiabatic eq. of state.
- Solve HSE + Poisson to get structure of polytrope:
  - ↳ $-\nabla\Psi = \frac{1}{\rho}\nabla(K\rho^{1+1/n}) = (n+1)\nabla(K\rho^{1/n})$

    $\Rightarrow$ $\rho = \left(\frac{\Psi_T-\Psi}{(n+1)K}\right)^n$, where $\Psi_T\equiv\Psi$ when $\rho=0$ (tidal potential)
  - ↳ the central density: $\rho_c = \left(\frac{\Psi_T-\Psi_c}{(n+1)K}\right)^n$ $\Rightarrow$ $\rho=\rho_c\left(\frac{\Psi_T-\Psi}{\Psi_T-\Psi_c}\right)^n$
  - ↳ let $\theta\equiv\frac{\Psi_T-\Psi}{\Psi_T-\Psi_c}$ and use a dimensionless radial coordinate ξ
  - ↳ gives the Lane-Emden equation: $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n$
  - ↳ at $\xi=0$ (centre), $\theta=1$ and $\frac{d\theta}{d\xi}=0$
- Lane-Emden can be solved for $n=0$, $n=1$, $n=5$.
- The limit $n\to\infty$ gives the isothermal sphere: $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\Psi}{d\xi}\right)=e^{-\Psi}$
  - ↳ $\rho\propto r^{-2}$ as $r\to\infty$, so mass does not converge.
  - ↳ in practice, we truncate at a finite radius → Bonnor-Ebert spheres

Scaling relations → c.f. stellar Homology

- Consider families of stars with the same polytrope index. The shape of the density curve in each star will be the same
- Using the rescaled coordinates:

  $\rho = \left(\frac{\Psi_T-\Psi}{(n+1)K}\right)^n$ $\Rightarrow$ $\Psi_T-\Psi_c = K(n+1)\rho_c^{1/n}$

  $\therefore\ \xi = r\sqrt{\frac{4\pi G\rho_c}{\Psi_T-\Psi_c}} = r\sqrt{\frac{4\pi G\rho_c^{1-1/n}}{K(n+1)}}$, $\quad\rho=\rho_c\theta^n$
  - ↳ at the surface, $\xi=\xi_{max}$, $\theta(\xi_{max})=0$
- The mass of a polytrope is:

  $$M=\int_0^{r_{max}}4\pi r^2\rho\,dr = 4\pi\rho_c\left[\frac{4\pi G\rho_c^{1-1/n}}{K(1+n)}\right]^{-3/2}\underbrace{\int_0^{\xi_{max}}\theta^n\xi^2\,d\xi}_{\text{same for a given } n}$$

  $\Rightarrow$ $M\propto\rho_c^{\frac{1}{2}(\frac{3}{n}-1)}$
  - ↳ radius relation comes from def. of ξ: $r_{max}\propto\rho_c^{1/2(\frac{1}{n}-1)}$
  - ↳ eliminate ρ_c to get the mass-radius relation: $M\propto R^{\frac{3-n}{1-n}}$
- For white dwarfs, $\gamma=5/3$ $\Rightarrow$ $n=3/2$, so $R\propto M^{-1/3}$. More massive WDs are smaller.
- However, $R\propto M^{-1/3}$ is wrong for most stars, because it assumes K is constant: $\left.\begin{aligned}p&=K\rho^{1+1/n}\\ p&=\frac{R^*}{\mu}\rho T\end{aligned}\right\}$ $\Rightarrow$ $T_c=\frac{\mu K}{R_*}\rho_c^{1/n}$
  - ↳ T_c is similar in all stars (nuclear reactions) $\Rightarrow$ $K\propto\rho_c^{-1/n}$

    $\Rightarrow$ $M\propto\rho_c^{-1/2}$, $R\propto\rho_c^{-1/2}$ $\Rightarrow$ $M\propto R$ as observed
  - ↳ we can use $K=\text{const}$ on an individual star when mass is changing on a fast timescale (before thermal eq. is established).

Sound Waves

Equilibrium Fluid

$\rho = \rho_0$ } uniform & constant

$p = p_0$

$\underline{u} = 0$

Small perturbation (Lagrangian)

$\rho = \rho_0 + \Delta\rho$

$p = p_0 + \Delta p$

$\underline{u} = \Delta\underline{u}$

- Eulerian perturbation: $\delta Q = \Delta Q - (\underline{\xi}\cdot\nabla)\,Q$ ← element of displacement
  - ↳ same as Lagrangian for uniform medium
- Apply perturbation to continuity/momentum eq. (to 1st order)
  - ↳ continuity ⇒ $\frac{\partial}{\partial t}\Delta\rho + \rho_0 \nabla\cdot(\Delta\underline{u}) = 0$
  - ↳ momentum ⇒ $\frac{\partial}{\partial t}(\Delta\underline{u}) = -\frac{1}{\rho_0}\nabla(\Delta p) = -\left.\frac{dp}{d\rho}\right|_{\rho_0}\frac{\nabla(\Delta\rho)}{\rho_0}$ } barotropic Eq of state
  - ↳ combine linearised eqs: $\boxed{\frac{\partial^2(\Delta\rho)}{\partial t^2} = \left.\frac{dp}{d\rho}\right|_{\rho_0}\cdot\nabla^2(\Delta\rho)}$ ← wave eq.
- Guess plane wave solution $\Delta\rho = \Delta\rho_0 e^{i(\underline{k}\cdot\underline{x} - \omega t)}$
  - ↳ $\omega^2 = \left.\frac{dp}{d\rho}\right|_{\rho_0} k^2$ ⇒ dispersionless waves
  - ↳ speed of sound: $\boxed{c_s = \sqrt{\left.\frac{dp}{d\rho}\right|_{\rho_0}}}$
- Can relate density ↔ velocity perturbations:
  - ↳ sub. $\Delta u, \Delta\rho$ into continuity eq ⇒ $-i\omega\Delta\rho + \rho_0 ik\Delta u = 0$

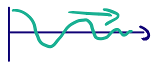
    ⇒ $\Delta u = \frac{\omega}{k}\frac{\Delta\rho}{\rho_0} = c_s\frac{\Delta\rho}{\rho_0}$
  - ↳ velocity and density perturbs. in phase
  - ↳ $\Delta u \ll c_s$, i.e sound wave much faster than fluid.

Sound waves in a stratified atmosphere

- E.g propagation through an isothermal atmosphere in HSE:
  - ↳ $u_0 = 0$, $\rho_0(z) = \tilde{\rho}e^{-z/H}$, $p_0(z) = \tilde{p}e^{-z/H}$, $H = \frac{R^*T}{g\mu}$
  - ↳ sound waves in the z direction are different
- With $\underline{g} = -g\hat{\underline{z}}$, momentum eq. becomes $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$
  - ↳ substitute Eulerian perturbations then convert to Lagrangian
  - ↳ continuity ⇒ $\frac{\partial\Delta\rho}{\partial t} + \rho_0\frac{\partial\Delta u_z}{\partial z} = 0$ (same as before)
  - ↳ momentum ⇒ $\frac{\partial\Delta u_z}{\partial t} = -\frac{c_u^2}{\rho_0}\frac{\partial\Delta\rho}{\partial z}$, $c_u \equiv \left.\frac{\partial p}{\partial\rho}\right|_{\rho_0}$
  - ↳ combine to give: $\underbrace{\frac{\partial^2\Delta\rho}{\partial t^2} - c_u^2\frac{\partial^2\Delta\rho}{\partial z^2}}_{\text{wave eq}} + \frac{c_u^2}{\rho_0}\underbrace{\frac{\partial\rho_0}{\partial z}}_{=-\frac{\rho_0}{H}}\frac{\partial\Delta\rho}{\partial z} = 0$
- Guessing plane wave solution: $\omega^2 = c_u^2\left(k^2 - \frac{ik}{H}\right)$ ← dispersion relation

$$k = \frac{i}{2H} \pm \sqrt{\frac{\omega^2}{c_u^2} - \frac{1}{4H^2}}$$

Case 1: $\omega > c_u/2H$



↳ $\Delta\rho \propto e^{-z/2H}e^{i(\mathrm{Re}(k) - \omega t)}$

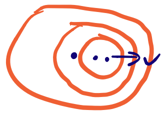
↳ $\frac{\Delta\rho}{\rho_0} \propto e^{z/2H}$, so perturbation theory fails eventually

Case 2: $\omega < c_u/2H$

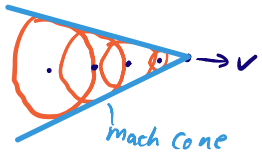
↳ $\Delta\rho \propto e^{-kz}e^{i\omega t}$ ← evanescent wave

Shocks

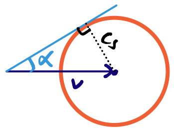
- In an isotropic medium, sound wavefronts are circular
  - ↳ for a moving source, the centres of subsequent wavefronts are displaced



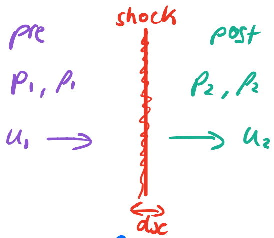
  - ↳ if $v > c_s$, a Mach cone forms:



  - ↳ the cone separates disturbed & undisturbed with a shockwave
  - ↳ the Mach number is $M \equiv \frac{v}{c_s}$; it determines the shape of the cone: $\sin\alpha = c_s/v = 1/M$



- For supersonic flows, the fluid may travel faster than signals can be transmitted → leads to discontinuities when the bulk 'realises' it has collided with something → shocks.
- Work in the reference frame of the shock.



- Integrate fluid equations over a small volume dx to get the Rankine-Hugoniot relations
- Continuity: $\frac{\partial}{\partial t}\left(\int_{0^-}^{0^+} \rho\, dx\right) = \rho u|_{0^+} - \rho u|_{0^-}$
  - ↳ in steady state, mass does not accumulate at $x=0$ $\Rightarrow \frac{\partial}{\partial t}[\cdot] = 0$

$$\Rightarrow \boxed{\rho_1 u_1 = \rho_2 u_2} \leftarrow 1^{st}\text{ R-H relation}$$

- Momentum: $\boxed{\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2} \leftarrow 2^{nd}$ R-H
- Energy equation for an adiabatic shock:

$$\frac{\partial E}{\partial t} + \nabla\cdot[(E+p)\underline{u}] = 0 \quad \Rightarrow (E_1+p_1)u_1 = (E_2+p_2)u_2$$

↳ $E = \rho(\frac{1}{2}u^2 + \varepsilon + \Phi) \Rightarrow \boxed{\frac{1}{2}u_1^2 + \varepsilon_1 + \frac{p_1}{\rho_1} = \frac{1}{2}u_2^2 + \varepsilon_2 + \frac{p_2}{\rho_2}} \leftarrow 3^{rd}$ R-H

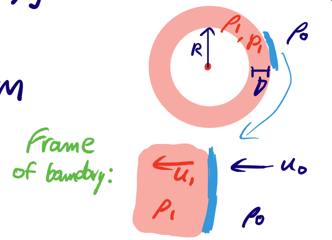
↳ can rewrite R-H III in terms of sound speed

$$\varepsilon = \frac{1}{\gamma-1}\frac{p}{\rho}, \quad c_s^2 = \frac{\gamma p}{\rho} \quad \Rightarrow \quad \frac{1}{2}u_1^2 + \frac{c_{s,1}^2}{\gamma-1} = \frac{1}{2}u_2^2 + \frac{c_{s,2}^2}{\gamma-1}$$

- Jumps in ρ, p, T can all be written in terms of γ and M
  - ↳ for strong shocks $(M \gg 1)$, $\frac{\rho_2}{\rho_1} \rightarrow \frac{\gamma+1}{\gamma-1} = \text{const}$
  - ↳ i.e. there is a maximum density jump for adiabatic shocks.
- 2^{nd} law of thermo. dictates the direction of the jump
  - ↳ Flow decelerates from super → sub-sonic, KE dissipated.
  - ↳ shocks are irreversible due to viscous processes

Supernova explosions

- Model supernova as point explosion of energy E within an ISM of density ρ_0, and $p_0 = 0$, $T_0 = 0$.
- Creates an expanding layer of shocked ISM



  - ↳ Strong shock $\Rightarrow \rho_1 = \rho_0 \frac{\gamma+1}{\gamma-1}$
  - ↳ shell consists of swept-up mass

$$\frac{4}{3}\pi R^3 \rho_0 = 4\pi R^2 D \rho_1 \Rightarrow D = \frac{1}{3}\frac{\gamma-1}{\gamma+1} R \simeq 0.08R$$

  - ↳ relative velocity in shell: $U = u_0 - u_1 = u_0 - \frac{\rho_0}{\rho_1}u_0 = \frac{2u_0}{\gamma+1}$
- The momentum of the shell is changing as it consumes ISM.
  - ↳ rate of chg of momentum: $\frac{d}{dt}\left[\frac{4}{3}\pi R^3 \rho_0 \cdot \frac{2u_0}{\gamma+1}\right]$
  - ↳ caused by pressure inside the cavity. Assume $p_{in} = \alpha p_1$
  - ↳ R-H II gives $p_1 = \frac{2}{\gamma+1}\rho_0 u_0^2$ ← Ram pressure into ISM

$$\Rightarrow \frac{d}{dt}\left[\frac{4}{3}\pi R^3 \rho_0 \cdot \frac{2u_0}{\gamma+1}\right] = 4\pi R^2 p_{in} = 4\pi R^2 \alpha \cdot \frac{2}{\gamma+1}\rho_0 u_0^2$$

$$\Rightarrow \frac{d}{dt}\left[R^3 u_0\right] = 3\alpha R^2 u_0^2 \Rightarrow \frac{d}{dt}\left[R^3 \dot{R}\right] = 3\alpha R^2 \dot{R}^2 \leftarrow u_0 \equiv \dot{R}$$

↳ seek power law solution $R \propto t^b \Rightarrow b = \frac{1}{4-3\alpha}$

↳ $R \propto t^{1/4-3\alpha} \Rightarrow u_0 = \dot{R} \propto R^{3\alpha-3}$

- Determine α by cons energy.
  - ↳ ignore KE of cavity (little mass) and internal energy of shell (thin)
  - ↳ KE of shell: $\frac{1}{2}\,\frac{4}{3}\pi R^3 \rho_0 u^2$
  - ↳ internal energy of cavity: $\frac{4}{3}\pi R^3 \rho\mathcal{E} = \frac{4}{3}\pi R^3 \alpha \frac{p_1}{\gamma-1}$
  - ↳ sum to get $E \propto R^3 u_0^2 \propto t^{(6\alpha-3)/(4-3\alpha)}$
  - ↳ E must be time dependent $\Rightarrow 6\alpha - 3 = 0 \Rightarrow \alpha = 1/2$
- Resulting dynamics: $\boxed{R \propto t^{2/5},\ u_0 \propto t^{-3/5},\ p_1 \propto t^{-6/5}}$

Similarity solutions to supernova explosions

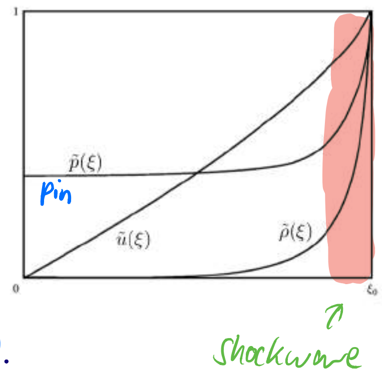
- Previous derivation assumed: uniform shell, $p_{in} \propto p_1$, cold ISM.
- Similarity solutions use dimensional analysis.
- We only specify E and ρ_0
  - ↳ unique combination to get a length scale: $\lambda = \left(\frac{Et^2}{\rho_0}\right)^{1/5}$
  - ↳ define dimensionless distance param $\xi \equiv \frac{r}{\lambda}$
  - ↳ the evolution of any variable in space and time can be separated into time behaviour $\times$ scale: $\boxed{X(t,r) = X_1(t)\tilde{X}(\xi)}$
- Can rewrite derivatives:

$$\frac{\partial X}{\partial r} = X_1 \frac{d\tilde{X}}{d\xi}\frac{\partial \xi}{\partial r}\Big|_t, \quad \frac{\partial X}{\partial t} = \tilde{X}(\xi)\frac{dX_1}{dt} + X_1 \frac{d\tilde{X}}{d\xi}\frac{\partial \xi}{\partial t}\Big|_r$$

↳ use $\rho(r,t) = X_\rho(t)\tilde{\rho}(\xi)$ etc and sub. into fluid equations

↳ fluid equations become ODEs

- Result is $R_{shock} \propto \left(\frac{E}{\rho_0}\right)^{1/5} t^{2/5}$
  - ↳ most of the mass is indeed swept into a shell
  - ↳ shell pressure is indeed a multiple of p_{in}
  - ↳ can take weighted avg of different shell velocities using the form of $\tilde{u}(\xi)$.



- The SN explosion stalls when $p_1 \sim p_0$:

$$p_1 = \frac{2}{\gamma+1}\rho_0 u_0^2, \quad c_s^2 = \frac{\gamma p_0}{\rho_0} \Rightarrow \frac{2}{\gamma+1}\rho_0 u_0^2 \sim \frac{\rho_0 c_s^2}{\gamma}$$

$$\Rightarrow u_0 \simeq c_s$$

↳ shell no longer supersonic → becomes a sound wave

↳ equivalently, when the energy of the explosion = internal energy swept up by the shockwave

Bernoulli's Equation

- Momentum equation: $\frac{\partial u}{\partial t} + (u\cdot\nabla)u = -\frac{1}{\rho}\nabla p - \nabla\Phi$
  - ↳ define the vorticity $w = \nabla\times u \Rightarrow (u\cdot\nabla)u = \nabla(\frac{1}{2}u^2) - u\times w$
  - ↳ for a barotropic fluid, $p = p(\rho)$

$$\frac{\partial}{\partial x}\int\frac{dp}{\rho} = \frac{\partial \rho}{\partial x}\frac{d}{d\rho}\int\frac{dp}{\rho} = \frac{1}{\rho}\frac{\partial p}{\partial x} \Rightarrow -\frac{1}{\rho}\nabla p = \nabla\left(\int\frac{dp}{\rho}\right)$$

$$\Rightarrow \frac{\partial u}{\partial t} + \nabla\left(\frac{1}{2}u^2\right) - u\times w = -\nabla\left(\int\frac{dp}{\rho} + \Phi\right)$$

  - ↳ for a steady flow, we then have

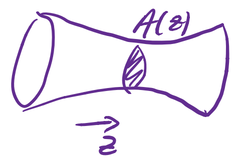
$$u\cdot\nabla\left(\frac{1}{2}u^2 + \int\frac{dp}{\rho} + \Phi\right) = 0$$

- Bernoulli's principle: $\boxed{H = \frac{1}{2}u^2 + \int\frac{dp}{\rho} + \Phi}$ is constant along a streamline (barotropic, steady flow).
- For a general barotropic (unsteady) flow: ← without viscosity

$$\frac{\partial u}{\partial t} = -\nabla H + u\times w \xrightarrow{\nabla\times} \boxed{\frac{\partial w}{\partial t} = \nabla\times(u\times w)}$$ ← Helmholtz Equation

  - ↳ the flux of vorticity through a surface S that moves with the fluid is constant: $\frac{D}{Dt}\int_S w\cdot dS = 0$ Kelvin's vorticity theorem
  - ↳ close analogy to magnetic field lines
- If $w = 0$, the fluid is irrotational
  - ↳ if irrotational, H is constant everywhere (not just on streamline)
  - ↳ if $w = 0$ it remains so $\Rightarrow u = -\nabla\Phi$
  - ↳ if flow is also incompressible, $\nabla\cdot u = 0 \Rightarrow \nabla^2\Phi = 0$

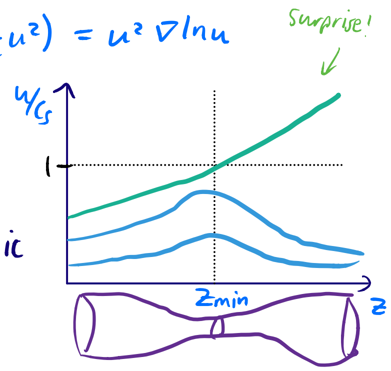
De Laval Nozzle



- Steady state barotropic flow through a nozzle
- Momentum: $u\cdot\nabla u = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho}c_s^2\nabla\rho$
- Continuity: $\rho u A = \text{const} \equiv \dot{m}$ } take logs then ∇
  - ↳ $\frac{1}{\rho}\nabla\rho = -\nabla\ln u - \nabla\ln A$
  - ↳ $u\cdot\nabla u = [\nabla\ln u + \nabla\ln A]c_s^2$
- For irrotational flow, $u\cdot\nabla u = \nabla(\frac{1}{2}u^2) = u^2\nabla\ln u$

$$\Rightarrow \boxed{(u^2 - c_s^2)\nabla\ln u = c_s^2\nabla\ln A}$$

- An extremum in A implies either
  - ↳ extremum in u
  - ↳ $u = c_s$, i.e subsonic → supersonic
  - ↳ fluid continues to accelerate

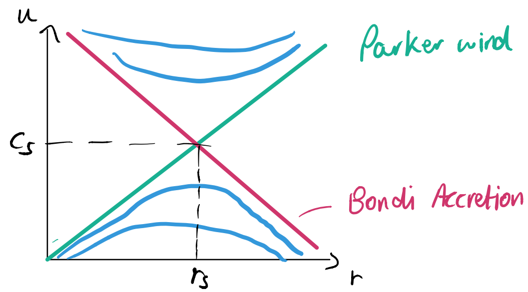


- For isothermal EoS, $p = \frac{R^*\rho T}{\mu} \Rightarrow H = \frac{1}{2}u^2 + c_s^2\ln\rho$
  - ↳ use Bernoulli equation ($H = \text{const}$) in terms of the min area A_m
  - ↳ $u^2 = c_s^2\left[1 + 2\ln\left(\frac{uA}{c_s A_m}\right)\right]$, using $\rho u A = \text{const}$
- For a polytropic EoS, c_s varies with density
  - ↳ $p = K\rho^{1+1/n} \Rightarrow c_s^2 = \frac{n+1}{n}K\rho^{1/n}$
  - ↳ $H = \frac{1}{2}\left(\frac{\dot{m}}{A\rho}\right)^2 + nc_s^2$

Spherical accretion

- Consider the spherically symmetric flow of matter onto a point mass. Assume steady state and barotropic EoS
- Continuity: $4\pi r^2 \rho u = \dot{M}$, where u points inwards
  - ↳ in a steady flow, $\frac{d}{dr}(\ln \dot{M}) = 0 \Rightarrow \frac{d}{dr}\ln\rho = -\frac{d}{dr}\ln u - \frac{2}{r}$
- Momentum: $u\frac{du}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM}{r^2} \Rightarrow u^2\frac{d\ln u}{dr} = -c_s^2\frac{d\ln\rho}{dr} - \frac{GM}{r^2}$
  - ↳ combine with continuity to get $\boxed{(u^2 - c_s^2)\frac{d}{dr}\ln u = \frac{2c_s^2}{r}\left(1 - \frac{GM}{2c_s^2 r}\right)}$ (*)
- There is a critical radius $r = r_s = \frac{GM}{2c_s^2}$ ← sonic point
  - ↳ at $r = r_s$, either $u = c_s$ or u is extremised.
- For Isothermal EoS, $c_s = \sqrt{\frac{R_* T}{\mu}} = \text{const}$
  - ↳ $H = \frac{1}{2}u^2 + c_s^2 \ln\rho + \Psi = \text{const}$ (Bernoulli)
  - ↳ compare flow at general point to flow at sonic point
    $\Rightarrow u^2 = 2c_s^2\left[\ln\left(\frac{\rho_s}{\rho}\right) - \frac{3}{2}\right] + 2\frac{GM}{r}$
  - ↳ as $r \to 0$, $u^2 \to 2GM/r$ (Free fall)
  - ↳ as $r \to \infty$ and $u \to 0$, $\rho \to \rho_s e^{-3/2} \Rightarrow \rho_s = \rho_\infty e^{3/2}$
  - ↳ mass accretion rate: $\dot{M} = 4\pi r_s^2 \rho_s c_s = \frac{\pi G^2 M^2 \rho_\infty e^{3/2}}{c_s^3}$
- For Polytropic EoS we repeat the same procedure except substitute $u = \frac{\dot{M}}{4\pi r^2 \rho}$ earlier to simplify
  $\Rightarrow \boxed{\dot{M} = \frac{\pi (GM)^2 \rho_\infty}{c_\infty^3}\left(\frac{n}{n - \frac{3}{2}}\right)^{n-3/2}}$ Bondi accretion
  - ↳ in $n \to \infty$ limit, becomes isothermal
  - ↳ for $n \to \frac{3}{2}$ ($\gamma = 5/3$), $\dot{M}$ still finite even though c_s, ρ singular.

- $\boxed{\dot{M} = AM^2} \Rightarrow M = \frac{M_0}{1 - At/M_0}$
  - ↳ $M \to \infty$ at finite time
  - ↳ in reality, accretion will be limited by fuel supply or the Eddington limit ($\dot{M} \propto M$)
- Dependence on reservoir properties:
  - ↳ $\boxed{\dot{M} \propto \frac{\rho_\infty}{c_\infty^3} \propto \frac{\rho_\infty}{c_\infty^5}}$ → higher accretion rates from colder material
  - ↳ For a moving accretion point with velocity v_∞ rel. to medium:
    $\dot{M} \sim \frac{(GM)^2 \rho_\infty}{(c_\infty^2 + v_\infty^2)^{3/2}}$ ← Bondi-Hoyle-Lyttleton
- Another solution to (*) is the Parker wind
  - ↳ physically, the very hot central gas causes an outward wind



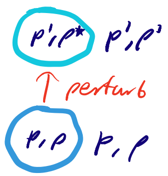
- These solutions are very sensitive to assumptions, e.g. nonzero angular momentum / B-fields break symmetry.

Robert Andrew Martin 2021

Fluid Instabilities

- A fluid is unstable if a perturbation to the steady state flow grows with time.
  - ↳ linearly unstable if an arbitrarily small perturbation grows
  - ↳ overstable if perturbations oscillate with growing amplitude
- Stable if perturbation decays/oscillates

Convective instability



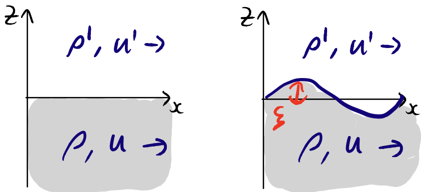
- Perturb a fluid element upwards (originally in HSE)
- Pressure will quickly equilibriate (acoustic waves), but there may not be time for heat exchange
  - ↳ density evolves adiabatically.
  - ↳ if $\rho^* < \rho'$, the perturbed element is buoyant and continues to rise → unstable.
- Adiabatic density change: $p = K\rho^\gamma$, $p' = K\rho^{*\gamma}$ $\Rightarrow \rho^* = \rho\left(\frac{p_1}{p}\right)^{1/\gamma}$
  - ↳ $p' = p + \frac{dp}{dz}\delta z \Rightarrow \rho^* = \rho\left(1 + \frac{1}{p}\frac{dp}{dz}\delta z\right)^{1/\gamma} \approx \rho + \frac{\rho}{p\gamma}\frac{dp}{dz}\delta z$
  - ↳ density of background atmosphere: $\rho' = \rho + \frac{d\rho}{dz}\delta z$
  - ↳ unstable if $\rho^* < \rho' \Rightarrow \frac{d}{dz}(\ln p\rho^{-\gamma}) < 0 \Rightarrow \boxed{\frac{dK}{dz} < 0}$
  - ↳ Schwarzchild criterion: convective unstable if entropy decreases upwards
  - ↳ temperature: $K \propto \rho^{1-\gamma}T^\gamma \Rightarrow \frac{dT}{dz} < \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{dp}{dz}$

- A convectively stable fluid undergoes SHM:
  - ↳ $\rho^* \frac{d^2}{dt^2}\delta z = -g(\rho^* - \rho')$
    $\Rightarrow \frac{d^2}{dt^2}\delta z = -\frac{g}{T}\left[\frac{dT}{dz} - \left(1 - \frac{1}{\gamma}\right)\frac{T}{p}\frac{dp}{dz}\right]\delta z$
  - ↳ internal gravity waves oscillating at the Brunt-Väisälä frequency

Gravitational instability

- Equilibrium: $p = p_0 = \text{const}$, $\rho = \rho_0 = \text{const}$, $\Psi = \Psi_0 = \text{const}$, $\underline{u} = 0$
  - ↳ Jeans swindle: technically can't have $\rho = \text{const}$ AND $\Psi = \text{const}$
- Governing equations: continuity, momentum, Poisson, barotropic EoS
- Perturb, e.g. $\rho = \rho_0 + \Delta\rho$, $\Psi = \Psi_0 + \Delta\Psi$
- Linearise and assume plane waves: $\Rightarrow \omega^2 = c_s^2\left(k^2 - \frac{4\pi G\rho_0}{c_s^2}\right)$
  - ↳ define Jeans wavenumber $k_J^2 = 4\pi G\rho_0/c_s^2 \Rightarrow \boxed{\omega^2 = c_s^2(k^2 - k_J^2)}$
  - ↳ $k \gg k_J \Rightarrow$ normal soundwaves
  - ↳ $k \gtrsim k_J \Rightarrow$ modified sound waves (slower group velocity)
  - ↳ $k < k_J \Rightarrow \omega$ imaginary so perturbations grow exponentially $\Rightarrow$ gravitational instability
- Maximum stable wavelength is the Jeans length $\lambda_J = \sqrt{\frac{\pi c_s^2}{G\rho_0}}$ with an associated Jeans mass $M_J \sim \rho_0\lambda_J^3$
  - ↳ systems undergo gravitational collapse when $M > M_J$
  - ↳ for isothermal collapse, $M_J \propto c_s^3\rho_0^{-1/2} \propto (T^3/\rho_0)^{1/2}$ so $M_J \downarrow$ as system collapses $\Rightarrow$ gravitational fragmentation.

Interface instabilities



- Interfaces have discontinuous changes in density / tangential velocity
  - ↳ assume incompressible and irrotational
    $\nabla\cdot\underline{u} = 0, \quad \nabla\times\underline{u} = 0 \Rightarrow u = -\nabla\Phi$
  - ↳ Φ is a velocity potential satisfying $\nabla^2\Phi = 0$
  - ↳ split potentials into perturbed and unperturbed
    $$\Phi_{low} = -Ux + \phi$$
    $$\Phi_{up} = -U'x + \phi' \quad \Rightarrow \nabla^2\phi = \nabla^2\phi' = 0$$
  - ↳ seek plane wave solutions
    $$\xi = A\exp(i(kx-\omega t))$$
    $$\phi = C\exp(i(kx-\omega t)+k_z z)$$
    $$\phi' = C'\exp(i(kx-\omega t)+k_z' z)$$
  - ↳ $\nabla^2\phi = 0$ and $\phi\to 0$ as $z\to-\infty \Rightarrow k_z = k$
  - ↳ similarly, $\nabla^2\phi' = 0 \Rightarrow k_z = -k$
- At the interface, $u_z = \frac{D\xi}{Dt} \Rightarrow -\frac{\partial\Phi}{\partial z} = \frac{\partial\xi}{\partial t} + U\frac{\partial\xi}{\partial x}$
  $$-\frac{\partial\Phi'}{\partial z} = \frac{\partial\xi}{\partial t} + U'\frac{\partial\xi}{\partial x}$$
  - ↳ sub in plane wave solution $\therefore -kC = i(kU-\omega)A$
    $$kC' = i(kU'-\omega)A$$
- Momentum equation $\Rightarrow \nabla\Big[\underbrace{-\frac{\partial\Phi}{\partial t} + \frac{1}{2}u^2 + \frac{p}{\rho} + \Psi}_{\hookrightarrow \text{ must be } F(t)}\Big] = 0$
  - ↳ pressure is continuous at the interface
    $$\therefore \rho\left(-\frac{\partial\Phi}{\partial t} + \frac{1}{2}u^2 + g\xi\right) = \rho'\left(-\frac{\partial\Phi'}{\partial t} + \frac{1}{2}u'^2 + g\xi\right) + \rho F(t) - \rho' F'(t)$$
  - ↳ $\rho F(t) - \rho' F'(t) \equiv K = \text{const}$ ← consider values at ∞

- Combine equations to get the dispersion relation:
  $$\rho(kU-\omega)^2 + \rho'(kU'-\omega)^2 = kg(\rho-\rho')$$
- For surface gravity waves, the denser fluid is below
  - ↳ $\rho' < \rho$ and let $U = U' = 0$ (fluids at rest)
  - ↳ dispersion relation: $\omega^2 = k\,\frac{g(\rho-\rho')}{\rho+\rho'} \Rightarrow \frac{\omega}{k} = f(k)$
  - ↳ if $\rho' \ll \rho$ (e.g. ocean), $\frac{\omega}{k} = \pm\sqrt{g/k}$
- If the denser fluid is on top (static):
  $$\omega^2 = k\,\frac{g(\rho-\rho')}{\rho+\rho'} \Rightarrow \frac{\omega}{k} = \pm i\sqrt{\frac{g}{k}\frac{\rho'-\rho}{\rho+\rho'}}$$
  - ↳ for $k\in\mathbb{R}$, ω is imaginary so there are exponentially growing/decaying solutions
  - ↳ Rayleigh-Taylor instability
- If the denser fluid is below but fluids are moving:
  - ↳ solve dispersion relation for $\frac{\omega}{k}$ ← quadratic
    $$\Rightarrow \frac{\omega}{k} = \frac{\rho U + \rho' U'}{\rho+\rho'} \pm \sqrt{\frac{g}{k}\frac{\rho-\rho'}{\rho+\rho'} - \frac{\rho\rho'(U-U')^2}{(\rho+\rho')^2}}$$
  - ↳ stability depends on sign inside square root
  - ↳ if $g = 0$, always unstable → Kelvin-Helmholtz instability
  - ↳ for $g \neq 0$, longer wavelengths are stabilised.

Thermal instability

- Consider perturbations of the energy equation
  - ↳ first rewrite in terms of K

$$p = K\rho^{\gamma} = \frac{R_*}{\mu}\rho T$$

$$\Rightarrow dK = \rho^{1-\gamma}(1-\gamma)\underbrace{\left[\frac{p}{\rho^2}d\rho + \frac{R_*}{\mu(1-\gamma)}dT\right]}_{=-dQ}$$

$$\Rightarrow \frac{DK}{Dt} = -(\gamma-1)\rho^{1-\gamma}\dot{Q}$$

  - ↳ gives the entropy form of the energy equation

$$\boxed{\frac{1}{K}\frac{DK}{Dt} = -(\gamma-1)\frac{\rho\dot{Q}}{p}}$$

- Assume the fluid is a static ideal gas (no gravity) in thermal equilibrium: $\underline{u}_0 = 0$, $\dot{Q}_0 = 0$, $\nabla K_0 = 0$
- Perturb and linearise:

$$\frac{\partial \Delta\rho}{\partial t} + \rho_0 \nabla\cdot(\Delta \underline{u}) = 0$$

$$\rho_0\frac{\partial \Delta u}{\partial t} = -\nabla(\Delta p)$$

$$\frac{\partial \Delta K}{\partial t} = \frac{-\gamma-1}{\rho_0^{\gamma-1}}\Delta\dot{Q} \leftarrow \text{sub } \Delta\dot{Q} = \frac{\partial\dot{Q}}{\partial p}\bigg|_{\rho}\Delta p + \frac{\partial\dot{Q}}{\partial\rho}\bigg|_{p}\Delta\rho$$

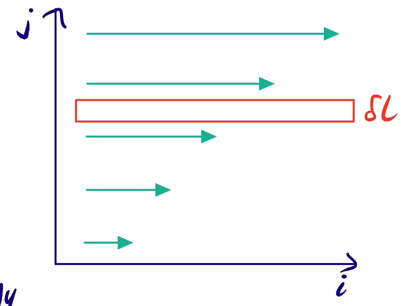
  - ↳ seek plane wave solutions e.g $\Delta\rho = \rho_1 e^{i\underline{k}\cdot\underline{x} + qt}$, such that $Re(q) > 0 \Rightarrow$ instability
  - ↳ result is a cubic dispersion relation $E(q) = 0$
  - ↳ unstable if the real root of $E(q)$ is > 0
  - ↳ Field criterion: unstable if $\boxed{\frac{\partial\dot{Q}}{\partial T}\bigg|_p < 0}$

☆ ↳ i.e unstable if cooling ↓ as temp ↑

  - ↳ e.g power-law cooling of the form $\dot{Q} \propto T^{\alpha}$ is unstable for $\alpha < 1$ (Bremsstrahlung has $\alpha = 0.5$)
- If a system is Field-unstable, all modes are unstable
- Even for Field-stable systems, there may be unstable modes
  - ↳ for large wavelengths (small K), $E(q) \approx q^2(q + A^*\rho_0^{\gamma})$
  - ↳ isochoric thermal instability: $\frac{\partial\dot{Q}}{\partial T}\bigg|_{\rho} < 0$ ← density, not pressure
  - ↳ for short wavelengths, sound waves bring pressure equilibrium so behaviour at const p matters
  - ↳ for long wavelengths, there is insufficient time for pressure to equalise → const ρ matters
- If gravity is included, buoyancy can stabilise thermal instabilities.

Viscous Flows

- We have assumed that changes in momentum are entirely due to pressure and gravity (valid for $\lambda \to 0$ limit) ← mean free path
- For finite λ, momentum can diffuse through the fluid
- Consider a linear shear flow
  - ↳ in addition to bulk flow, there are random thermal velocities
  - ↳ flux of i momentum in j direction



$$\boxed{\langle \rho v_i v_j \rangle = \underbrace{\alpha \rho u_i}_{\text{bulk momentum density}} \underbrace{\sqrt{\frac{k_B T}{m}}}_{\text{thermal velocity in } j \text{ direction}}}$$

  - ↳ α is a constant of order 1. For hard spheres, $\alpha = \frac{5\sqrt{\pi}}{64}$
  - ↳ the net momentum flux through a plane of thickness δl is: $-\rho(\partial_j u_i)\delta l \alpha \sqrt{\frac{kT}{m}}$
  - ↳ $\delta l \approx \lambda = \frac{1}{n\sigma} = \frac{1}{n\sigma} = \frac{m}{\rho\sigma} = \frac{m}{\rho\pi a^2}$

    $\Rightarrow$ momentum flux $= -(\partial_j u_i)\frac{m}{\sigma} \alpha \sqrt{\frac{kT}{m}}$

- This momentum flux modifies the momentum equation:

$$\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij}) + \partial_j\left[\frac{\alpha}{\sigma}\sqrt{mkT}\,\partial_j u_i\right] + \rho g_i$$

  - ↳ $\boxed{\eta \equiv \frac{\alpha}{\sigma}\sqrt{mkT}}$ is the shear viscosity
  - ↳ η is independent of density → more particles, but shorter λ
  - ↳ $\eta \uparrow$ with T. Isothermal systems have $\eta = \text{const}$
  - ↳ For Coulomb interactions, $\lambda \propto T^2 \Rightarrow \eta \propto T^{5/2}$

Navier-Stokes

- We can generalise to allow for viscous stresses in different directions
- Define the viscous stress tensor σ'_{ij}. It must be:
  1. Invariant to Galilean transformations
  2. Linear in velocity gradients
  3. Isotropic
  - ↳ the most general tensor that satisfies this is

$$\sigma'_{ij} = \eta\underbrace{\left(\partial_j u_i + \partial_i u_j - \tfrac{2}{3}\delta_{ij}\partial_k u_k\right)}_{\text{shear flow}} + \underbrace{\zeta}_{\text{bulk viscosity}} \underbrace{\delta_{ij}\partial_k u_k}_{\text{compression}}$$

  - ↳ momentum equation: $\frac{\partial}{\partial t}(\rho u_i) = -\partial_j(\rho u_i u_j + p\delta_{ij} + \sigma'_{ij}) + \rho g_i$
  - ↳ combining this with continuity gives the Navier-Stokes equation
- For an isothermal unshocked fluid, $\eta = \text{const}$ and $\zeta \approx 0$

$$\Rightarrow \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Psi + \frac{\eta}{\rho}\left[\nabla^2\underline{u} + \frac{1}{3}\nabla(\nabla\cdot\underline{u})\right]$$

  - ↳ $\frac{\eta}{\rho} \equiv \nu$ is the coefficient of kinematic viscosity
- The importance of viscosity is characterised by the Reynolds number

$$\boxed{Re = \frac{|\underline{u}\cdot\nabla\underline{u}|}{|\nu\nabla^2\underline{u}|} \sim \frac{UL}{\nu}}$$

U is a velocity scale, L is a lengthscale

  - ↳ viscosity important for small Re
- Viscosity allows for momentum transmission by shearing:
  - ↳ stabilises fluid instabilities (i.e. damping)
  - ↳ irreversibly dissipates KE as heat

- Vorticity: take the curl of the Navier-Stokes equation

$$\Rightarrow \boxed{\frac{\partial \underline{w}}{\partial t} = \nabla \times (\underline{u} \times \underline{w}) + \frac{\eta}{\rho} \nabla^2 \underline{w}}$$

↳ viscosity allows for vorticity to diffuse through the fluid

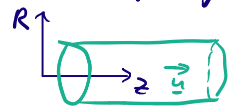
↳ relative importance of advection/diffusion is given by Re

↳ vorticity can be introduced into an irrotational flow due to boundary interactions → then diffuses into the bulk

I.4

Flow in a pipe

- Consider a steady-state, laminar, incompressible flow through a circular pipe (neglecting gravity)



- Neglect edge effects ⇒ $u = u(R)$
- Navier-Stokes: $\underbrace{\cancel{\frac{\partial \underline{u}}{\partial t}}}_{\text{steady}} + \underbrace{\cancel{\underline{u} \cdot \nabla \underline{u}}}_{\text{symmetry}} = -\frac{1}{\rho} \nabla p - \underbrace{\cancel{\nabla \psi}}_{\text{no gravity}} + \nu \left(\nabla^2 \underline{u} + \underbrace{\cancel{\tfrac{1}{3} \nabla (\nabla \cdot \underline{u})}}_{\text{incompressible}} \right)$

$$\Rightarrow \nu \nabla^2 \underline{u} = \frac{1}{\rho} \nabla p$$

↳ laminar ⇒ $u_R = u_\phi = 0$ ⇒ $\frac{\partial p}{\partial R} = \frac{\partial p}{\partial \phi} = 0$

↳ z-component: $\underbrace{\nu \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial u_z}{\partial R} \right)}_{R \text{ only}} = \underbrace{\frac{1}{\rho} \frac{\partial p}{\partial z}}_{z \text{ only}} = \text{const} = -\frac{1}{\rho} \frac{\Delta p}{L}$

- Integrate equation to get $u = -\frac{\Delta p}{4\rho\nu L} R^2 + a \ln R + b$

↳ finite at $R = 0$ ⇒ $a = 0$

↳ $u(R_0) = 0$ (no slip)

$$\Rightarrow u = \frac{\Delta p}{4\nu\rho L} (R_0^2 - R^2)$$

- Mass flux: $\boxed{Q = \int_0^{R_0} 2\pi R \rho u \, dR}$
- Beyond a certain Re (flow rate), there will be turbulence.

Accretion

- If infalling gas has net angular momentum (almost always true), Bondi accretion is not applicable. Gas tends to form accretion disks $\Rightarrow$ near-keplerian orbits
- Accretion requires fluid elements to lose angular momentum
- Model a thin accretion disk in cylindrical coordinates:
  - ↳ axisymmetry $\Rightarrow$ $\partial/\partial\phi = 0$
  - ↳ HSE vertically $\Rightarrow$ $u_z = 0$
- Angular velocity: $\Omega = \sqrt{\frac{GM}{R^3}}$ $\rightarrow$ varies with R so there will be shear between layers.
- Continuity: $\frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) + \underbrace{\cancel{\frac{1}{R}\frac{\partial}{\partial\phi}(\rho u_\phi)}}_{\text{axisymmetry}} + \underbrace{\cancel{\frac{\partial}{\partial z}(\rho u_z)}}_{\text{HSE}} = 0$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\rho u_R) = 0$$

  - ↳ For accretion disks, the surface density is more relevant $\Sigma \equiv \int_{-\infty}^{\infty} \rho\, dz \Rightarrow \frac{\partial\Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_R) = 0$
  - ↳ can also derive directly by considering flow of mass in/out of annulus
- Likewise for the momentum equation, we can use Navier-Stokes or consider annuli:
  - ↳ $\frac{d}{dt}\left(\begin{smallmatrix}\text{angular}\\ \text{momentum}\end{smallmatrix}\right) = \text{ang mtm in} - \text{ang mtm out} + \text{net torque}$
  - $\Rightarrow \frac{\partial}{\partial t}(2\pi R\Delta R\,\Sigma\,\Omega R^2) = F(R) - F(R+\Delta R) + G(R+\Delta R) - G(R)$
  - ↳ $F(R) \equiv 2\pi R\Sigma u_R \Omega R^2$ describes the advection of ang mtm
  - ↳ viscous torque: $\boxed{G(R) = 2\pi R\,\nu\Sigma R\frac{\partial\Omega}{\partial R} R = 2\pi R^3 \nu\Sigma\frac{d\Omega}{dR}}$

$$\Rightarrow \frac{\partial}{\partial t}(R\Sigma u_\phi) = -\frac{1}{R}\frac{\partial}{\partial R}(\Sigma R^2 u_\phi u_R) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{d\Omega}{dR}\right)$$

- For an axisymmetric orbit, $\frac{\partial u_\phi}{\partial t} = 0$

$$\Rightarrow u_R = \frac{\frac{\partial}{\partial R}\left(\nu\Sigma R^3\frac{d\Omega}{dR}\right)}{R\Sigma\frac{\partial}{\partial R}(R^2\Omega)}$$

  - ↳ sub u_R into continuity eq and use Newtonian point source with $\Omega = \sqrt{GM/R^3}$ $\Rightarrow$ $\boxed{\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}\left(\nu\Sigma R^{1/2}\right)\right]}$
  - ↳ surface density obeys a diffusion-like equation
  - ↳ if $\nu = \nu(R)$ only, it becomes a linear diffusion equation
  - ↳ hence narrow rings start to spread out, becoming disks.
- Estimate accretion time:

$$\frac{\Sigma}{t_\nu} \sim \frac{1}{R}\cdot\frac{1}{R}\left[R^{1/2}\frac{1}{R}\nu\Sigma R^{1/2}\right] \sim \frac{\nu\Sigma}{R^2}$$

$$\Rightarrow t_\nu \sim \frac{R^2}{\nu} = \frac{R}{u_\phi}\,\frac{Ru_\phi}{\nu} = \Omega^{-1}\cdot Re$$

  - ↳ kinetic theory gives very small ν $\rightarrow$ accretion timescale greater than the age of the universe
  - ↳ i.e the 'viscosity' driving accretion cannot be microphysical viscosity $\rightarrow$ believed to be due to turbulence.
  - ↳ $[\nu] = [L]^2[T]^{-1} = [u][L]$. Using the characteristic quantities in a disk, $\nu = \alpha c_s H$, where $\alpha < 1 = \text{const}$

Steady-state accretion disks

- Energy dissipation per unit area of disk

flux → $F_{disk} = -\int \sigma_{ij}' \partial_j u_j u_i \frac{dV}{2\pi R dR d\phi}$

$$= \frac{1}{2}\int \eta (\partial_j u_i + \partial_i u_j)^2 dz$$

$$= \int \eta R^2 \left(\frac{d\Omega}{dR}\right)^2 dz = \nu \Sigma R^2 \left(\frac{d\Omega}{dR}\right)^2$$

- Continuity: $\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u_r) = 0$

↳ steady state ⇒ $R\Sigma u_r = C_1 = -\frac{\dot{m}}{2\pi}$ ← $\dot{m}$ positive for influx

⇒ $u_R = -\frac{\dot{m}}{2\pi R \Sigma}$

- Axisymmetric orbit: $u_R = \frac{\frac{\partial}{\partial R}\left(\nu \Sigma R^3 \frac{d\Omega}{dR}\right)}{R\Sigma \frac{\partial}{\partial R}(R^2\Omega)}$

↳ steady state accretion around a point mass $\left(\Omega^2 = \frac{GM}{R^3}\right)$

⇒ $-\frac{\dot{m}}{2\pi R\Sigma} = -\frac{3}{\Sigma R^{1/2}}\frac{\partial}{\partial R}\left(\nu \Sigma R^{1/2}\right)$

↳ integrate with inner torque boundary condition → no viscous torque at some inner radius i.e $\nu\Sigma = 0$ at $R = R_*$

⇒ $\nu \Sigma = \frac{\dot{m}}{3\pi}\left(1 - \sqrt{\frac{R_*}{R}}\right)$

- Sub into dissipation formula: $\boxed{F_{diss} = \frac{3GM\dot{m}}{4\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right)}$

↳ total energy emitted : $L = \int_{R_*}^{\infty} F_{diss} \cdot 2\pi R dR = \frac{GM\dot{m}}{2R_*}$

↳ i.e disk radiates half of the accreting matter's binding energy

↳ other half is KE of infalling matter

- Far from the inner disk, $F_{diss} \approx \frac{3GM\dot{m}}{4\pi R^3}$

↳ the local rate of loss of binding energy is:

$$F_{diss,est} = \underbrace{\frac{1}{2\pi R dR}}_{\text{area of annulus}} \cdot \underbrace{\left|\frac{\partial}{\partial R}\left(\frac{GM\dot{m}}{R}\right)\right|}_{\Delta GPE} \cdot \underbrace{\frac{1}{2}}_{1/2 \text{ radiated}} = \frac{GM\dot{m}}{4\pi R^3}$$

↳ i.e 2/3 of dissipated energy is not from ΔGPE

↳ source is viscous transport of energy from inner → outer disk

- Can estimate disk temp assuming BB radiation:

↳ top & bottom → $2 \cdot \sigma T_{eff}^4 = \frac{3GM\dot{m}}{4\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right)$

↳ at large distances, $\boxed{T_{eff} \propto R^{-3/4}}$

- All observables are independent of viscosity → need to observe non-steady disks to learn about ν.

Plasmas

Magnetohydrodynamics (MHD)

- Model fully ionised hydrogen as two cohabiting fluids:
  1. Proton fluid $m^+, n^+, \underline{u}^+$
  2. Electron fluid $m^-, n^-, \underline{u}^-$
- Aggregate properties of fluid:
  - ↳ density $\rho = m^+n^+ + m^-n^-$
  - ↳ COM velocity is the density-weighted avg: $\underline{u} = \frac{m^+n^+\underline{u}^+ + m^-n^-\underline{u}^-}{\rho}$
  - ↳ charge density $q = n^+e^+ + n^-e^-$
  - ↳ current density $\underline{j} = e^+n^+\underline{u}^+ + e^-n^-\underline{u}^-$
- Conserve particle number: $\frac{\partial n^\pm}{\partial t} + \nabla\cdot(n^\pm \underline{u}^\pm) = 0$
  - ↳ multiply eqns by $m^\pm$ to get $\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$ ← standard continuity
  - ↳ multiply by $e^\pm$ to get charge conservation: $\frac{\partial q}{\partial t} + \nabla\cdot\underline{j} = 0$
- Momentum equation:
  - ↳ Lorentz force on each particle: $F = q(\underline{E} + \underline{u}\times\underline{B})$
  - ↳ for each fluid: fraction of pressure gradient attributed to each fluid

$$m^\pm n^\pm\left(\frac{\partial \underline{u}^\pm}{\partial t} + \underline{u}^\pm\cdot\nabla\underline{u}^\pm\right) = e^\pm n^\pm(\underline{E} + \underline{u}^\pm\times\underline{B}) - f^\pm\nabla p$$

  - ↳ sum: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + q\underline{E} + \underline{j}\times\underline{B}$ new terms
- Ohm's law connects $\underline{j}$ with the electromagnetic fields:
  $\underline{j} = \sigma(\underline{E} + \underline{u}\times\underline{B})$ where σ is the conductivity.

- Equations of MHD:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0 \qquad \text{Mass continuity}$$

$$\frac{\partial q}{\partial t} + \nabla\cdot\underline{j} = 0 \qquad \text{Charge continuity}$$

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + q\underline{E} + \underline{j}\times\underline{B} \qquad \text{Momentum}$$

$$\underline{j} = \sigma(\underline{E} + \underline{u}\times\underline{B}) \qquad \text{Ohm's law}$$

+ Maxwell's equations

Ideal MHD

- Consider a non relativistic and highly conducting plasma
- Approx fields as varying over lengthscale l and timescale τ.
- $\nabla\times\underline{E} = -\frac{\partial \underline{B}}{\partial t} \Rightarrow \frac{E}{l} \sim \frac{B}{\tau} \Rightarrow \frac{E}{B} \sim u$
- $\frac{\left|\frac{1}{c^2}\frac{\partial \underline{E}}{\partial t}\right|}{|\nabla\times\underline{B}|} \sim \frac{1}{c^2}\left(\frac{l}{\tau}\right)^2 \sim \frac{u^2}{c^2} \ll 1$ (non-relativistic)
  - ↳ can ignore displacement current $\Rightarrow \nabla\times\underline{B} = \mu_0\underline{j}$
- $\frac{|q\underline{E}|}{|\underline{j}\times\underline{B}|} \sim \frac{qE}{jB} \sim \frac{\varepsilon_0(\nabla\cdot\underline{E})E}{\frac{1}{\mu_0}|\nabla\times\underline{B}|B} \sim \frac{\varepsilon_0 E/L\, E}{\frac{1}{\mu_0}B/L\, B} \sim \frac{u^2}{c^2} \ll 1$
  - ↳ $q\underline{E}$ is negligible in the momentum eq.
  - ↳ i.e charge neutrality is preserved
- Combine Ampere/Ohm: $\nabla\times\underline{B} = \mu_0\underline{j} = \mu_0\sigma(\underline{E} + \underline{u}\times\underline{B})$
  - ↳ take curl $\Rightarrow \frac{\partial \underline{B}}{\partial t} = \nabla\times(\underline{u}\times\underline{B}) + \frac{1}{\mu_0\sigma}\nabla^2\underline{B}$ ← same form as vorticity
  - ↳ good conductor $\Rightarrow \sigma$ large $\Rightarrow \boxed{\frac{\partial \underline{B}}{\partial t} = \nabla\times(\underline{u}\times\underline{B})}$

↳ magnetic flux is 'frozen' into the plasma

- For a good conductor, $\underline{E} + \underline{u}\times\underline{B} = \frac{1}{\sigma}\underline{j} \to 0 \Rightarrow \underline{E}\perp\underline{B}$
- Ideal momentum equation: $\boxed{\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \frac{1}{\mu_0}(\nabla\times\underline{B})\times\underline{B}}$

↳ electromagnetic force per unit volume is $\underline{f}_{mag} = \frac{1}{\mu_0}(\nabla\times\underline{B})\times\underline{B}$

↳ using vector identity: $\underline{f}_{mag} = \frac{1}{\mu_0}\left[-\nabla\underbrace{\left(\frac{B^2}{2}\right)}_{\text{magnetic pressure}} + \underbrace{(\underline{B}\cdot\nabla)\underline{B}}_{\text{magnetic tension}}\right]$

↳ can absorb magnetic pressure into ∇p to get:

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = \frac{1}{\mu_0}(\underline{B}\cdot\nabla)\underline{B} - \nabla p_{tot} \leftarrow p_{tot} = p + \frac{B^2}{2\mu_0}$$

MHD waves

- New terms in the momentum eq → different waves.
- Perturb density, pressure, fluid velocity, B-field ← assume barotropic EoS
- Seek plane wave solutions, e.g. $\Delta\underline{B} = \Delta\underline{B}_0 e^{i(\underline{k}\cdot\underline{r} - \omega t)}$

↳ $\frac{\partial}{\partial t} \to -i\omega$, $\nabla \to i\underline{k}$

↳ continuity → $\omega\Delta\rho = \rho_0\underline{k}\cdot\Delta\underline{u}$

↳ momentum → $\omega\rho_0\Delta\underline{u} = \frac{1}{\mu_0}[(\underline{B}_0\cdot\Delta\underline{B})\underline{k} - (\underline{B}_0\cdot\underline{k})\Delta\underline{B}] + c_s^2\Delta\rho\,\underline{k}$

↳ flux-freezing → $\omega\Delta\underline{B} = \underline{B}_0(\underline{k}\cdot\Delta\underline{u}) - (\underline{B}_0\cdot\underline{k})\Delta\underline{u}$

① Perturbation perpendicular to field, $\underline{k}\perp\underline{B}_0$

↳ simplify eqns then eliminate $\Delta\rho$, $\Delta\underline{B}$

↳ $\Delta\underline{u} \propto \underline{k}$ so this is a longitudinal mode

↳ result is $\omega^2 = \left(c_s^2 + \frac{B^2}{\mu_0\rho_0}\right)k^2$

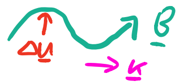
↳ define the Alfvén speed $\boxed{v_A = \sqrt{\frac{B^2}{\rho\mu_0}} \Rightarrow \omega^2 = (c_s^2 + v_A^2)k^2}$

↳ it is a fast magnetosonic wave, travelling faster than the sound speed due to magnetic pressure.

② Perturbation parallel to field, $\underline{k} \parallel \underline{B}_0$

↳ $\omega^2 = \frac{B_0^2}{\mu_0\rho_0}k^2 \Rightarrow \boxed{\omega^2 = v_A^2k^2}$ ← Alfvén waves

↳ transverse incompressible waves due to magnetic tension

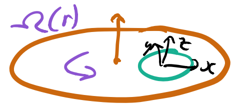


↳ another permitted solution is the standard sound wave

- For a general perturbation ($\underline{B}$, $\underline{k}$ at angle θ) there are 3 modes: Alfvén wave, fast magnetosonic, slow magnetosonic

Magneto-rotational instability

- Consider the local frame of a patch in an accretion disk



- Momentum eq:

$$\frac{D\underline{u}}{Dt} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{pressure gradient}} + \underbrace{\frac{1}{\mu_0\rho}(\nabla\times\underline{B})\times\underline{B}}_{\text{magnetic force}} + \underbrace{2\underline{u}\times\underline{\Omega}}_{\text{Coriolis}} + \underbrace{\underline{\Omega}\times(\underline{\Omega}\times\underline{r})}_{\text{Centrifugal}} - \underbrace{R\,\Omega_K(R)^2\,\hat{\underline{R}}}_{\text{gravity}}$$

- Assume:

↳ uniform field $\underline{B}_0 = B_0\hat{\underline{z}}$ ← aligned with $\underline{\Omega}$

↳ cold gas → ignore pressure

↳ only consider $\underline{k} \parallel \underline{B}_0$ perturbations

- Perturbed eqn: $\frac{D\Delta \underline{u}}{Dt} - 2\Delta\underline{u} \times \underline{\Omega} = \frac{1}{\mu_0 \rho}(\underline{B_0} \cdot \nabla)\Delta B - \underbrace{\Delta x\, R \frac{d\Omega^2}{dR}}_{\text{mismatch of centri/gravity}} \hat{\underline{R}}$
  - ↳ seek plane wave solutions and use MHD eqs to get expression for $\Delta \underline{B}$
  - ↳ result: $\omega^4 - \omega^2\left[4\Omega^2 - \frac{d\Omega^2}{d\ln R} + 2(kv_A)^2\right] + (kv_A)^2\left[(kv_A)^2 + \frac{d\Omega^2}{d\ln R}\right] = 0$
- Ignoring magnetic physics: $\omega^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R} = \frac{1}{R^3}\frac{d}{dR}(R^4\Omega^2) \equiv \kappa_R^2$
  - ↳ if $\kappa_R^2 > 0$ we get radial epicyclic approximations
  - ↳ if $\kappa_R^2 < 0$ (i.e h decreases with radius), the flow is unstable.
- Including magnetism, unstable if $\omega^2 < 0 \Rightarrow (kv_A)^2 + \frac{d\Omega^2}{d\ln R} < 0$
  - ↳ if the field is weak, kv_A is negligible
    - $\Rightarrow$ unstable if $\frac{d\Omega^2}{dR} < 0$
  - ↳ hence even Keplerian flow is unstable → *magnetorotational instability*.
  - ↳ MRI is stabilised if $k > k_{crit}$:
    - $(k_{crit} v_A)^2 = -\frac{d\Omega^2}{d\ln R} = 3\Omega^2$ in Keplerian case