Relativity

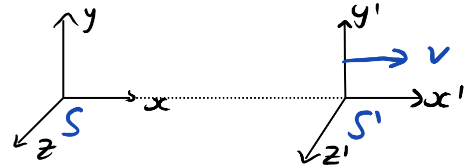
- There are several different masses in Newtonian gravity
  - ↳ passive gravitational, m_G, which experiences a force in a field $\vec{F} = -m_G \nabla\phi$
  - ↳ active gravitational, m_A, which generates the field according to Poisson's eq: $\nabla^2\phi = 4\pi G\rho$
  - ↳ inertial mass, m_I, where $\vec{F} = m_I \ddot{x}$
- $m_G = m_A$ by NIII; this is also the case with charge in electrodynamics.
- However, the equality of m_G and m_I is an experimental fact. It means that particles of any mass accelerate at the same rate in response to a grav. field.
  - ↳ weak equivalence principle: free falling particles follow the same path.
  - ↳ certainly not true for electromag.
- The strong equivalence principle states that a uniformly accelerating frame is indistinguishable from a frame experiencing gravitation (so we can apply SR).
  - ↳ constant grav fields are unobservable
  - ↳ inertial frames should be defined w.r.t free falling obervers.

Special Relativity

- Newtonian gravity inconsistent with SR because it assumes ϕ changes instantaneously as ρ changes
- Inertial frames are those for which free particles obey NI: $\ddot{\underline{x}} = \underline{0}$
- The principle of relativity states that physics is the same in every inertial frame (this is a special case of free falling frames).

Transforming between frames

- The standard configuration:



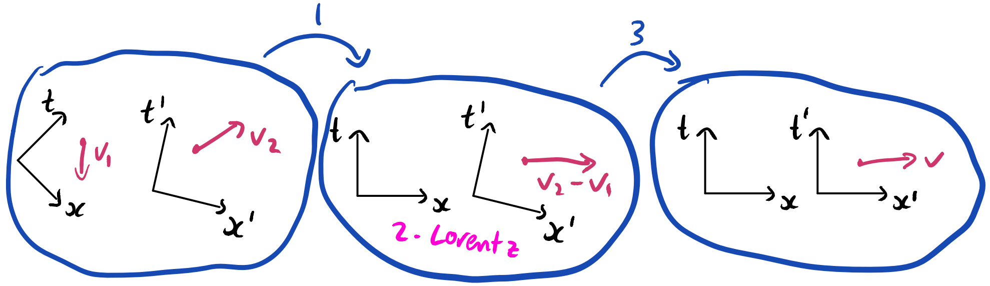
- Event coordinates in S and S' are related by a linear transformation. Symmetry restrictions and $x'=0 \Rightarrow x = vt$, $x=0 \Rightarrow x' = -vt$ result in:

$$t' = At + Bx \qquad x' = A(x - vt).$$

- Newtonian mechanics assumes absolute time: $t' = t$ so $x' = x - vt$. This Galilean transformation implies:
  - ↳ $\Delta t = t_B - t_A$ for events A, B is invariant
  - ↳ $\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ is invariant for simult. events
- SR replaces absolute time with a different postulate: the invariance of c (based on the principle of relativity)

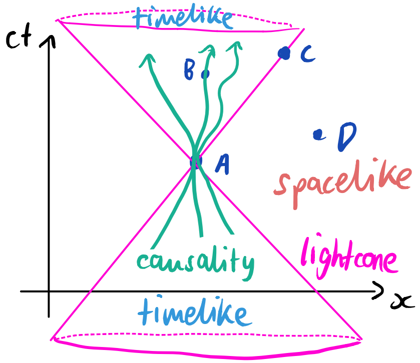
- For a photon emitted from the coincident origin of S, S' at $t = t' = 0$, speed = dist/time implies:
  $$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0$$
  ↳ can be subbed into general linear transform to derive the Lorentz transform:
  $$\boxed{ct' = \gamma(ct - \beta x)\,;\quad x' = \gamma(x - \beta ct) \quad \text{where } \beta \equiv v/c \text{ and } \gamma \equiv (1-\beta^2)^{-1/2}}$$
  ↳ we can define the interval, invariant under Lorentz boosts: $\Delta s^2 = c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$
  ↳ this invariance defines Minkowski spacetime
- Lorentz boosts can be viewed as 4D 'rotations':
  ↳ $\beta \in [-1, 1]$ so we can define the rapidity ψ such that $\beta = \tanh\psi$
  $$\Rightarrow \gamma = \cosh\psi, \quad \gamma\beta = \sinh\psi$$
  $$\therefore \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\psi & -\sinh\psi \\ -\sinh\psi & \cosh\psi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$
  ↳ invariance of Δs^2 now follows from trig identities.
- General Lorentz boosts can be expressed in terms of the standard config:
  1. Rotate S to align x-axis with the relative velocity
  2. Lorentz boost, resulting in a frame S'' comoving w/ S'
  3. Spatially rotate $S'' \to S'$.

  (Lorentz boosts are easy whenever axes are aligned)

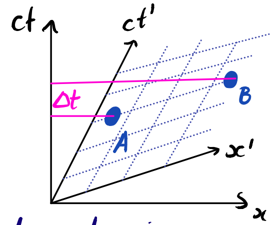


Light cones and simultaneity

- The sign of Δs^2 allows us to classify event separations:
  ↳ $\Delta s^2 > 0 \rightarrow$ timelike
  ↳ $\Delta s^2 = 0 \rightarrow$ lightlike
  ↳ $\Delta s^2 < 0 \rightarrow$ spacelike
- Events outside A's lightcone cannot affect / be affected by A.



- For timelike intervals we can find an inertial frame in which the events occur at the same spatial coords
- For spacelike intervals we can find an inertial frame in which the events are simultaneous.
  ↳ $x' = 0 \Rightarrow x = \beta ct$
  ↳ $ct' = 0 \Rightarrow ct = \beta x$
  ↳ A, B simultaneous in S' but not S



- This implies that simultaneity is not Lorentz invariant.

- However, the temporal ordering of events is invariant as long as the interval is timelike or lightlike, i.e
  $c\Delta t > 0 \Rightarrow c\Delta t' > 0$ if $\Delta s \geq 0$

Length contraction & time dilation

- Consider a rod of proper length l_0 at rest in S' (S, S' in the standard config), so that $l_0 = x_B' - x_A'$
  ↳ an observer in S measures a contracted length
  $\Delta x' = \gamma(\Delta x - v\Delta t)$ and $\Delta t = 0$
  $\Rightarrow l = \Delta x = \frac{1}{\gamma}\Delta x' = l_0/\gamma$
- Consider a clock at rest in S' with period T_0:
  $c\Delta t = \gamma(c\Delta t' + \beta\Delta x')$ and $\Delta x' = 0$
  $\Rightarrow T = \gamma T_0$

Paths in spacetime

- The interval between two events along a general path in spacetime is $\Delta s = \int_A^B ds$, where ds is the invariant Minkowski line element $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$
- A particle describes a worldline, in which each ds will be within the infinitesimal light cone.
  ↳ rather than describing the path as $(x(t), y(t), z(t))$, we parameterise with λ: $(t(\lambda), x(\lambda), y(\lambda), z(\lambda))$

↳ For a massive particle, it is convenient to use the proper time τ – time measured by ideal clock on particle.
↳ in an instantaneous rest frame (IRF) of the particle,
$dx' = dy' = dz' = 0 \Rightarrow ds^2 = c^2d\tau^2$
↳ ds^2 invariant $\Rightarrow \underbrace{c^2d\tau^2 = c^2dt^2 - dx^2 - dy^2 - dz^2}_{\text{General frame } S}$
$\Rightarrow \boxed{d\tau = \frac{1}{\gamma_v}dt}$
↳ γ_v is the Lorentz factor where v is the speed of the particle as seen in S
↳ the total proper time can be found by integrating
$$\Delta\tau = \int_A^B d\tau = \int_A^B \sqrt{1 - \frac{v(t)^2}{c^2}}\,dt$$

The Doppler effect

- Consider a signal with period $\Delta t'$ being emitted from the origin of S', moving away at speed v.
- In frame S, the photons travelled $c\gamma\Delta t'$ and the source travelled $v\gamma\Delta t'$ by the next flash. The time between received pulses is then
  $\Delta t = \frac{1}{c}(c\gamma\Delta t' + v\gamma\Delta t')$
  $\Rightarrow \frac{f}{f'} = \sqrt{\frac{1-\beta}{1+\beta}}$

Velocity addition

- Consider a particle on worldline $(x(t), y(t), z(t))$ in S, with velocity $u_x = \frac{dx}{dt}$, $u_y = \frac{dy}{dt}$, $u_z = \frac{dz}{dt}$. In standard config, with S' moving at v relative to S:

↳ $c\,dt' = \gamma_v(c\,dt - \beta x)$ $\qquad dx' = \gamma_v(dx - \beta c\,dt)$

$dy' = dy \qquad dz' = dz$

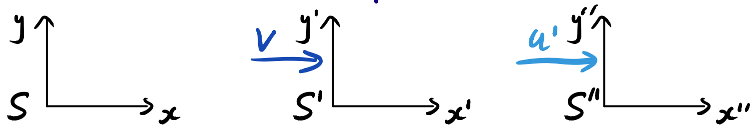
↳ the velocity in S' is then

$$u_x' = \frac{dx'}{dt'} = \frac{u_x - v}{1 - u_x v/c^2} \qquad u_y' = \frac{dy'}{dt'} = \frac{u_y}{\gamma_v(1 - u_x v/c^2)}$$

↳ inverse transform: swap primes, flip signs.

↳ reduces to Galilean transformation as $v \ll c$

- Alternatively, we can think of the particle as being at rest in S'' which moves at speed u' relative to S', while S' moves at speed v relative to S:



↳ $x'' = \cosh\psi_u\, x' - \sinh\psi_{u'}\, ct'$

$= \ldots$ write $S \to S'$ boost and use trig identities

$\therefore\ x'' = \cosh(\psi_v + \psi_{u'})x - \sinh(\psi_v + \psi_{u'})ct$

↳ likewise, $ct'' = \cosh(\psi_v + \psi_{u'})ct - \sinh(\psi_v + \psi_{u'})x$

↳ hence, the two colinear boosts are equivalent to a single boost with speed $u = c\tanh(\psi_v + \psi_{u'})$

↳ this can be expanded to give the particle speed u in S

Acceleration

$$a_x' = \frac{du_x'}{dt'} = \left(\frac{\partial u_x'}{\partial u_x}\frac{du_x}{dt} + \frac{\partial u_x'}{\partial u_y}\frac{du_y}{dt} + \frac{\partial u_x'}{\partial u_z}\frac{du_z}{dt}\right)\frac{dt}{dt'}$$

$$u_x' = \frac{u_x - v}{1 - u_x v/c^2} \Rightarrow \frac{\partial u_x'}{\partial u_x} = \frac{1 - v^2/c^2}{(1 - u_x v/c^2)^2} \qquad \to a_x \qquad dt' = \gamma_v\left(1 - \frac{u_x v}{c^2}\right)dt$$

$$\Rightarrow a_x' = \frac{a_x}{\gamma_v^3(1 - u_x v/c^2)^3}$$

- Acceleration is not invariant but is absolute – all observers agree on whether or not a particle is accelerating.
- Consider a particle moving in the x direction of S with speed $u(\tau)$ and a proper acceleration $f(\tau)$ in it's IRF (instantaneous rest)

↳ objective is to find worldline in S

↳ consider a frame S'

$$u' = \frac{u - v}{1 - uv/c^2} \Rightarrow \frac{du'}{d\tau} = \frac{1}{\gamma_v^2(1 - uv/c^2)^2}\frac{du}{d\tau}$$

↳ set S' to be the IRF at time τ: $v = u(\tau)$, $f(\tau) = \frac{du'}{d\tau}$

$$\Rightarrow f(\tau) = \frac{\gamma_u^4}{\gamma_u^2}\frac{du}{d\tau} \Rightarrow \frac{du}{d\tau} = \frac{f(\tau)}{\gamma_u^2}$$

↳ writing in terms of $u = c\tanh\Psi$: $\Psi(\tau) = \frac{1}{c}\int_0^\tau f(\tau')d\tau'$

↳ the worldline can then be recovered from $\Psi(\tau)$

$$\frac{dt}{d\tau} = \gamma_u = \cosh\Psi(\tau) \qquad \frac{dx}{d\tau} = u\gamma_u = c\sinh\Psi(\tau)$$

· If the acceleration is constant, $\Psi(\tau) = a\tau/c$

$$\Rightarrow \quad ct(\tau) = ct_0 + \frac{c^2}{a}\sinh(a\tau/c)$$
$$x(\tau) = x_0 + \frac{c^2}{a}\left(\cosh(a\tau/c) - 1\right)$$

} defines a hyperbola

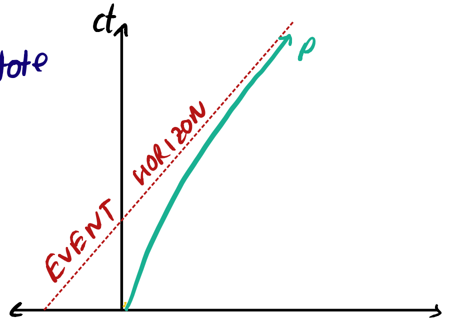
↳ as $\tau \to \infty$, $\sinh\tau \sim \cosh\tau$

so there is an oblique asymptote

$ct = \frac{c^2}{a} + x$

↳ events to the left of the event horizon cannot be seen by the particle

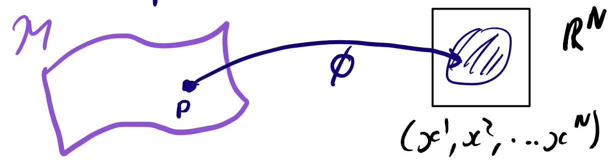
↳ other particles will not appear to cross the event horizon.



Manifolds and Coordinates

· A ND manifold is a set of objects that locally resemble $\mathbb{R}^N$. In GR, these objects are events.

↳ there exists a one-one map ϕ from manifold $\mathcal{M}$ to an open subset of $\mathbb{R}^N$



↳ we may need to stitch several euclidean spaces.

· Curves are parametrically defined : $x^a = x^a(u)$, $a = 1 \ldots N$

↳ more generally, an M-D surface needs M params

$$x^a = x^a(u^1, u^2, \ldots u^M)$$

↳ hypersurfaces have N-1 dimensions and can instead be specified as $f(x^1, x^2, \ldots, x^N) = 0$

· Coordinate transformations are passive relabellings :

$$x'^a = x'^a(x^1, x^2, \ldots x^N)$$

$$dx'^a = \sum_{b=1}^{N} \frac{\partial x'^a}{\partial x^b} dx^b$$

↳ there is an $N \times N$ transformation matrix at each point P

$$J^a{}_b = \frac{\partial x'^a}{\partial x^b} = \begin{pmatrix} \frac{\partial x'^1}{\partial x^1} & \cdots & \frac{\partial x'^1}{\partial x^N} \\ \vdots & \ddots & \vdots \\ \frac{\partial x'^N}{\partial x^1} & \cdots & \frac{\partial x'^N}{\partial x^N} \end{pmatrix}$$

a

b

↳ $J = \det(J^a{}_b)$ is the Jacobian. If $J \neq 0$, the coordinate transform is invertible.

Local geometry of Riemannian manifolds

- Local geometry is specified with an invariant distance.
- In a Riemannian manifold, this interval is quadratic in coordinate differentials: $\boxed{ds^2 = g_{ab}(x)\, dx^a dx^b}$ ← Einstein sums

↳ the metric functions g_{ab} are symmetric.

↳ to relabel coordinates, use interval invariance:

$$ds^2 = g_{ab}\, dx^a dx^b$$
$$= \underbrace{g_{ab} \frac{\partial x^a}{\partial x'^c} \frac{\partial x^b}{\partial x'^d}}_{g'_{cd}} \partial x'^c \partial x'^D$$

- Intrinsic geometry is fully specified by metric functions: an ant on the surface could determine it. Extrinsic geometry can only be appreciated by a higher-dimensional obs.

↳ the curved surface of a cylinder is intrinsically identical to $\mathbb{R}^2$: $ds^2 = a^2 d\phi^2 + dz^2$. $a\,d\phi = dx \Rightarrow ds^2 = dx^2 + dz^2$

↳ but on a sphere, there is no global substitution like above, though it may be locally Euclidean.

$$x^2 + y^2 + z^2 = a^2 \Rightarrow dz = -\frac{(x\,dx + y\,dy)}{\sqrt{a^2 - x^2 - y^2}}$$

$$\therefore ds^2 = dx^2 + dy^2 + \frac{(x\,dx + y\,dy)^2}{a^2 - (x^2 + y^2)}$$

↳ using plane polars, $ds^2 = \frac{a^2 d\rho^2}{(a^2 - \rho^2)} + \rho^2 d\phi^2$

- The length along path $x^a(u)$ is $\int_{u_A}^{u_B} \left| g_{ab} \frac{dx^a}{du} \frac{dx^b}{du} \right|^{1/2} du$
- For a diagonal metric, the coordinate system is orthogonal:

$$ds^2 = g_{11}(dx^1)^2 + \dots + g_{NN}(dx^N)^2$$

↳ the volume element is $dV = \sqrt{|g_{11} \dots g_{NN}|}\, dx^1 \dots dx^N$

↳ for general non-orthogonal systems, $\boxed{dV = \sqrt{|g|}\, dx^1 \dots dx^N}$

determinant

Local Cartesian coordinates

- Not possible in general to chose coordinates such that the line element is everywhere Euclidean.
- But at any point P we can chose coordinates s.t.

$$g_{ab}(P) = \delta_{ab}, \quad \left.\frac{\partial g_{ab}}{\partial x^c}\right|_P = 0 \quad \leftarrow \text{locally Cartesian}$$

↳ near P, $g_{ab}(x) = \delta_{ab} + O((x - x_P)^2)$

↳ we can show that there are enough d.o.f:

$g_{ab}(P) = \delta_{ab}$ requires $\frac{N(N+1)}{2}$ d.o.f

$g'_{ab} = \frac{\partial x^c}{\partial x'^a} \frac{\partial x^d}{\partial x'^b} g_{cd}$ has N^2, so $\frac{N(N-1)}{2}$ leftover.

↳ for the derivative condition:

$\left.\frac{\partial g_{ab}}{\partial x^c}\right|_P = 0$ requires $N^2(N+1)/2$ constraint eqs

$$\frac{\partial g'_{ab}}{\partial x'^e} = \frac{\partial}{\partial x'^e}\left(\frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\right) g_{cd} + \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\frac{\partial x^f}{\partial x'^e}\frac{\partial g_{cd}}{\partial x^f}$$

gives $N^2(N+1)/2$ d.o.f, so we can meet constraints

↳ however, we cannot set all second derivs by a coordinate transform → curvature of spacetime.

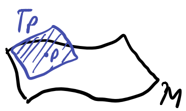
- In pseudo-Riemannian manifolds ds^2 can be ≤ 0.
  - ↳ we can always find local coordinates at P such that $g_{ab}(P) = \eta_{ab}$, $\left.\frac{\partial g_{ab}}{\partial x^c}\right|_P = 0$

$$\eta_{ab} = \mathrm{diag}(\pm 1, \pm 1, \dots, \pm 1)$$

  - ↳ the signature of a pseudo-Riemannian manifold is num positive less num negative entries in η_{ab}.

Vector algebra on manifolds

- Scalar fields assign a value to point P, indep. of coords.
- Displacement vectors do not generalise to manifolds, but infinitesimal vectors (e.g E fields) do:
  - ↳ the tangent space $T_P(M)$ of M at point P is an ND vector space whose elements are local displacement vectors.
  - ↳ clearly $T_P(M)$ is different at diff points.



- Vectors can be thought of as linear differential operators:

$$\underset{\sim}{V} = V^a \left.\frac{\partial}{\partial x^a}\right|_P$$

real components basis vectors

  - ↳ this fits all the conditions for a vector space
  - ↳ a vector field specifies a local vector $\underset{\sim}{V} \in T_P(M)$ at each point $P \in M$.
- Vectors themselves are invariant; only the labels change. For transformation from $x \to x'$:
  - ↳ basis vectors transform via the chain rule $\left.\frac{\partial}{\partial x'^a}\right|_P = \left.\frac{\partial x^b}{\partial x'^a}\frac{\partial}{\partial x^b}\right|_P$
  - ↳ for $\underset{\sim}{V}$ to be invariant, the components must transform oppositely: $V'^a = \left.\frac{\partial x'^a}{\partial x^b}\right|_P V^b$

$$\Rightarrow V'^a \left.\frac{\partial}{\partial x'^a}\right|_P = V^b\left(\frac{\partial x'^a}{\partial x^b}\frac{\partial x^c}{\partial x'^a}\frac{\partial}{\partial x^c}\right)\Big|_P = V^b \left.\frac{\partial}{\partial x^b}\right|_P = \underset{\sim}{V}$$

- The gradient of a scalar field is not a vector.
  - ↳ $X_a = \frac{\partial \phi}{\partial x^a}$. Under transform $x \to x'$:

$$X'_a = \frac{\partial \phi'}{\partial x'^a} = \frac{\partial \phi}{\partial x^b}\frac{\partial x^b}{\partial x'^a} = X_b \frac{\partial x^b}{\partial x'^a}$$

this is J^{-1}, not J as it is for a vector

  - ↳ objects that transform like this are dual vectors/covectors, and form the dual vector space $T^*_P(M)$
- The contraction $X_a V^a$ of a covector and vector gives a coordinate-independent quantity:

$$X'_a V'^a = \frac{\partial x^b}{\partial x'^a}\frac{\partial x'^a}{\partial x^c} X_b V^c = \delta_{bc} X_b V^c = X_b V^b$$

↳ in general there is no invariant way of associating covectors and vectors, unless we are on a Riemannian manifold with a metric.

↳ for orthogonal transforms ($\partial x'^a / \partial x^b$ is an orthogonal matrix), components of covectors/vectors transform the same way ($J^T = J^{-1}$) so for Cartesian coordinates there is no distinction.

Tensor fields

- Tensors of type (k,l) have k "upstairs" vector-like indices and l "downstairs" covector-like indices: $T^{a\ldots b}{}_{c\ldots d}$
  - ↳ tensors take k covectors and l vectors at P to return a scalar
  - ↳ the rank of a tensor is $k+l$.
  - ↳ tensor transformation:

$$T'^{a\ldots b}{}_{c\ldots d} = \frac{\partial x'^a}{\partial x^p} \cdots \frac{\partial x'^b}{\partial x^q} \frac{\partial x^r}{\partial x'^c} \cdots \frac{\partial x^s}{\partial x'^d} T^{p\ldots q}{}_{r\ldots s}$$

  - ↳ tensor fields assign a tensor (same type) to every $P \in M$
  - ↳ distinction between the invariant tensor $\underline{T}$ and its components $T^{a\ldots b}{}_{c\ldots d}$.
- Tensors allow us to write equations that work independently of a coordinate system.
- Rules for tensor operations:
  - ↳ add two tensors of the same type to give a new tensor of the same type. $\underline{T} + \underline{S}$: $T_{ab} + S_{ab}$
  - ↳ multiply by a real number $c\underline{T}$: cT_{ab}
  - ↳ for tensors S, type (p,q) and T, type (r,s), the tensor product $\underline{S} \otimes \underline{T}$ is type $(p+r, q+s)$.
  - ↳ tensor product not commutative in general
  - ↳ contraction turns type $(k,l) \to (k-1, l-1)$ by setting an upstairs index = downstairs index then summing.
- Tensors are symmetric if $S_{ab} = S_{ba}$, antisymmetric if $S_{ab} = -S_{ba}$
  - ↳ tensors can be decomposed into symmetric and antisymmetric parts (coordinate free): $S_{ab} = \underbrace{\tfrac{1}{2}(S_{ab}+S_{ba})}_{S_{(ab)}} + \underbrace{\tfrac{1}{2}(S_{ab}-S_{ba})}_{S_{[ab]}}$
  - ↳ extends to many indices (of the same type).
  - ↳ $S_{(ab\ldots c)}$ is totally symmetric under swapping any indices;

$$S_{(ab\ldots c)} = \frac{1}{k!}(\text{sum over perms of } ab\ldots c)$$

e.g. $S_{(abc)} = \frac{1}{6}(S_{abc} + S_{acb} + S_{cab} + S_{cba} + S_{bca} + S_{bac})$

  - ↳ $S_{[ab\ldots c]}$ is totally antisymmetric, changing sign for odd perms

$$S_{[ab\ldots c]} = \frac{1}{k!}(\text{alternating sum over perms of } ab\ldots c)$$

e.g. $S_{[abc]} = \frac{1}{6}(S_{abc} - S_{acb} + S_{bca} - S_{bac} + S_{cba} - S_{cab})$

- We can test if an object is a tensor by checking if it transforms as a tensor. The quotient theorem is a shortcut: if $X_{ab\ldots c}$ contracts with any tensor to form a new tensor, then $X_{ab\ldots c}$ are the components of a tensor.

The metric tensor

- The metric in a pseudo-Riemannian manifold transforms as a type-$(0,2)$ tensor: $ds^2 = g_{ab}dx^a dx^b = g'_{ab}dx'^a dx'^b$
  $$\Rightarrow g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b} g_{cd}$$
  ↳ we can think of the metric tensor as mapping two vectors to a real number (i.e an inner product):
  $$\underline{g}(\underline{u},\underline{v}) = g_{ab}u^a u^b$$
  ↳ contracting a vector V^a with the metric tensor gives a dual vector (coordinate-free): $V_a = g_{ab}V^b$

  ↳ more generally, we can change the type of a tensor (lowering an index) by contracting with the metric tensor e.g $T_{ab} = g_{ac}T^c{}_b$, or $T_{abc} = g_{ap}g_{bq}T^{pq}{}_c$
- The inverse metric is a type-$(2,0)$ tensor. Denote $(g^{-1})^{ab} \equiv g^{ab}$:

  ↳ by definition of inverse, $g^{ab}g_{bc} = \delta^a_c$
  $$\Rightarrow g'^{ab} = \frac{\partial x'^a}{\partial x^c}\frac{\partial x'^b}{\partial x^d} g^{cd}$$
  ↳ we can now get a vector from its dual (raising index), using $X^a = g^{ab}X_b$

  ↳ must preserve horizontal order of indices when raising/lowering
- The mixed components of the metric come from raising one index:

  ↳ $g^a{}_c = g^{ab}g_{bc} = \delta^a_c = g_c{}^a$

  ↳ $g^a{}_b$ is thus special because it is the only rank-2 tensor whose components are the same in all coordinate systems (always $= \delta^a_b$)
- The inner product can now be written in 3 ways:

  ↳ $g_{ab}u^a v^b = u_a v^a = u^a v_a$

  ↳ we define the invariant norm as $|\underline{v}| = \sqrt{|g_{ab}v^a v^b|}$

  ↳ orthogonal vectors have $\underline{g}(\underline{u},\underline{v}) = 0$

Tensor Calculus

Covariant derivatives

- The gradient of a scalar field is a dual vector $\nabla\phi$ with components $\partial\phi/\partial x^a$
  - ↳ contraction with an infinitesimal displacement gives $\delta\phi = \frac{\partial\phi}{\partial x^a}\delta x^a$
- For a tensor field, we want a derivative that is also a tensor
  - ↳ not as simple as for scalar fields. Consider vector field $v^a(x)$ and the derivative $\partial v^b/\partial x^a$

$$\frac{\partial v'^b}{\partial x'^a} = \frac{\partial}{\partial x'^a}\left(\frac{\partial x'^b}{\partial x^c}v^c\right) = \frac{\partial x^d}{\partial x'^a}\frac{\partial}{\partial x^d}\left(\frac{\partial x'^b}{\partial x^c}v^c\right)$$

$$= \underbrace{\frac{\partial x^d}{\partial x'^a}\frac{\partial x'^b}{\partial x^c}\frac{\partial v^c}{\partial x^d}}_{\text{tensor}} + \underbrace{\frac{\partial x^d}{\partial x'^a}\frac{\partial^2 x'^b}{\partial x^d \partial x^c}v^c}_{\text{not tensor}}$$

  - ↳ we must therefore construct a derivative that transforms like a tensor.
- The covariant derivative of a type-(k,l) tensor is a type-$(k,l+1)$ tensor $\nabla_c T^{a_1\dots a_k}{}_{b_1\dots b_l}$
  - ↳ we construct $\boxed{\nabla_a v^b = \partial_a v^b + \Gamma^b_{ac}v^c}$
  - ↳ Γ^b_{ac} are the connection coefficients (not a tensor), chosen to make $\partial_a v^b \equiv \partial v^b/\partial x^a$ transform like a tensor.
  - ↳ ∇_a acting on a scalar field is just the gradient $\nabla_a\phi = \partial_a\phi$
  - ↳ ∇_c is a linear operator; satisfies the product rule.
  - ↳ we impose that ∇_a commutes with contraction. Hence we can compute ∇_a on a covector field:

$$\nabla_a(X_b v^b) = (\nabla_a X_b)v^b + X_b\nabla_a v^b \text{ (Product) but } X_b v^b \text{ scalar}$$

$$\Rightarrow \nabla_a(X_b v^b) = \partial_a(X_b v^b) = \partial_a(X_b)v^b + X_b\,\partial_a v^b$$

$$\Rightarrow \boxed{\nabla_a X_b = \partial_a X_b - \Gamma^c_{ab}X_c}$$

  - ↳ covariant derivatives for general tensor fields can now be formed with the product rule.
  - ↳ $\nabla_c\, g^a{}_b = \partial_c\delta^a_b + \Gamma^a{}_{cd}\,\delta^d_b - \Gamma^d{}_{cb}\,\delta^a_d = 0$; equivalent to requiring ∇_a commutes with contraction.
- On a manifold with a metric, we enforce 2 conditions for ∇_a
  1. Metric compatibility: $\nabla_a g_{bc} = 0$ and $\nabla_a g^{bc} = 0$
  2. Commutative on scalar fields: $\nabla_a\nabla_b\phi = \nabla_b\nabla_a\phi$
  - ↳ ② ⇒ the connection is symmetric in lower indices: $\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$
  - ↳ ① ⇒ $0 = \nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d{}_{ca}\,g_{db} - \Gamma^d{}_{cb}\,g_{ad}$ and cyclic perms.
  - ↳ combine to give the Christoffel symbols (metric connection):

$$\boxed{\Gamma^a{}_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc}\right)}$$

- Covariant differentiation can be interchanged with raising or lowering indices: $\nabla_c T^{ab} = \nabla_c(g^{bd}T^a{}_d) = g^{bd}(\nabla_c T^a{}_d)$
  - ↳ we can thus treat gradients as vectors: $\nabla^a\phi = g^{ab}\nabla_b\phi$

↳ using Jacobi's formula for an invertible matrix $\underline{M}$
$|\underline{M}|^{-1}\partial_c|\underline{M}| = \mathrm{Tr}(\underline{M}^{-1}\partial_c \underline{M})$, combined with
$\nabla_c g_{ab} = 0 \Rightarrow g^{ab}\partial_c g_{ab} = 2g^{ab}g_{ab}\Gamma^{a}{}_{ca} = 2\Gamma^{a}{}_{ac}$, we get
the contraction of the connection:

$$\boxed{\Gamma^{a}{}_{ac} = \frac{1}{2}g^{-1}\partial_c g = |g|^{-1/2}\partial_c|g|}$$

- In local Cartesian coordinates, $g_{ab}(p) = \delta_{ab}$, $\partial_c g_{ab}|_p = 0$
  ↳ Christoffel symbols vanish at this point
  ↳ thus $\nabla_a v^b \to \partial_a v^b$. We could have instead derived an expr for ∇_a starting with local Cartesians
  ↳ this equivalence allows us to rewrite SR field equations using ∇_a instead of ∂_a, and the equations will be general.

- The divergence of a vector field is the scalar field $\nabla_a V^a$.
$$\nabla_a V^a = \partial_a V^a + \Gamma^{a}{}_{ab}V^b = |g|^{-1/2}\partial_a(|g|^{1/2}V^a)$$
- The curl of a dual-vector field is the antisymmetry of the covariant derivative (type-(0,2) tensor): $\boxed{(\mathrm{curl}\ X)_{ab} \equiv \nabla_a X_b - \nabla_b X_a}$
  ↳ by symmetry of connection $\mathrm{curl}\ X = \partial_a X_b - \partial_b X_a$
  ↳ the notion of curl as a vector does not generalise beyond 3D
- The Laplacian is the contraction of two covariant derivatives:
$$\nabla^2 T^{ab} = \nabla^c\nabla_c T^{ab} = g^{cd}\nabla_c\nabla_d T^{ab}.$$

e.g. Covariant derivatives on unit 2-sphere in $\mathbb{R}^3$

$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \Rightarrow g_{ab} = \mathrm{diag}(1, \sin^2\theta)$

$\Gamma^{a}{}_{bc} = \frac{1}{2}g^{ad}(\partial_b g_{dc} + \partial_c g_{db} - \partial_d g_{bc})$ ← because g_{ab} diagonal

$a=\theta$, sum over d : $2\Gamma^{\theta}{}_{bc} = g^{\theta\theta}(\partial_b g_{\theta c} + \partial_c g_{\theta b} - \partial_\theta g_{bc})$
$= (\partial_b\delta_{\theta c} + \partial_c\delta_{\theta b} - \partial_\theta g_{bc}) = -\partial_\theta g_{bc}$

Only non-zero connection component is $\Gamma^{\theta}{}_{\phi\phi} = -\frac{1}{2}\partial_\theta \sin^2\theta = -\sin\theta\cos\theta$

$a=\phi$: $2\Gamma^{\phi}{}_{bc} = g^{\phi\phi}(\partial_b g_{\phi c} + \partial_c g_{\phi b} - \partial_\phi g_{bc})$
$= \sin^{-2}\theta(\delta_{\phi c}\partial_b\sin^2\theta + \delta_{\phi b}\partial_c\sin^2\theta)$
$\therefore \Gamma^{\phi}{}_{\theta\phi} = \Gamma^{\phi}{}_{\phi\theta} = \frac{1}{2}\sin^{-2}\theta\,\partial_\theta\sin^2\theta = \cot\theta.$

$\nabla_a V^b = \partial_a V^b + \Gamma^{b}{}_{ac}V^c$
$\therefore\ \nabla_\theta V^\theta = \partial_\theta V^\theta$
$\nabla_\theta V^\phi = \partial_\theta V^\phi + \Gamma^{\phi}{}_{\theta c}V^c = \partial_\theta V^\phi + \cot\theta\, V^\phi$
$\nabla_\phi V^\theta = \partial_\phi V^\theta + \Gamma^{\theta}{}_{\phi c}V^c = \partial_\phi V^\theta - \sin\theta\cos\theta\, V^\phi$
$\nabla_\phi V^\phi = \partial_\phi V^\phi + \Gamma^{\phi}{}_{\phi c}V^c = \partial_\phi V^\phi + \cot\theta\, V^\theta$

Grad: $\nabla^a\psi = g^{ab}\nabla_b\psi = (g^{\theta\theta}\partial_\theta\psi, g^{\phi\phi}\partial_\phi\psi) = (\partial_\theta\psi, \sin^{-2}\theta\,\partial_\phi\psi)$
Div: $\nabla_a V^a = \partial_\theta V^\theta + \partial_\phi V^\phi + \cot\theta\, V^\theta$
Laplacian : $\nabla^2\psi = \nabla_a\nabla^a\psi = \partial_\theta\partial_\theta\psi + \partial_\phi(\sin^{-2}\theta\,\partial_\phi\psi) + \cot\theta\,\partial_\theta\psi$
$= \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$

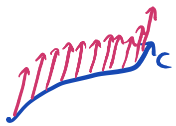
Intrinsic derivatives and parallel transport

- The intrinsic derivative is the projection of the covariant derivative onto the tangent to a curve.
- Given a vector $v^a(u)$ defined along a curve $x^a(u)$ (e.g the momentum at a point on the particle's worldline):

$$\boxed{\frac{Dv^a}{Du} \equiv \frac{dx^b}{du}\nabla_b v^a = \frac{dv^a}{du} + \frac{dx^b}{du}\Gamma^a_{bc}v^c}$$

ordinary derivatives!

  - ↳ contract covariant derivative with the tangent vector
  - ↳ definition applies to tensors
  - ↳ intrinsic deriv has same properties as covariant derivative (e.g linearity, Leibniz).
- In Cartesians, a vector is parallel-transported if the components are constant as we move along a curve i.e $\frac{dv^a}{du} = 0$



- Parallel transport on a general manifold is defined as $\boxed{Dv^a/Du = 0}$
  - ↳ generalises to tensors, e.g $DT^{ab}/Du = 0$.
  - ↳ given a curve, there will be a unique parallel-transported vector (given some ICs)
  - ↳ independent of parameterisation: $\delta v^a = -\delta x^b \Gamma^a_{bc} v^c$ ← connects vectors at diff points
  - ↳ scalar products (and thus lengths) are preserved under transport:

$$\frac{d(\underline{v}\cdot\underline{w})}{du} = \frac{D}{Du}\left(g_{ab}v^a w^b\right) = g_{ab}v^a\frac{Dw^b}{Du} + g_{ab}w^a\frac{Dv^b}{Du} = 0$$

  - ↳ parallel transport is path-dependent.

Geodesic curves

- Geodesics are the generalisations of straight lines to curved space
  - ↳ defined as the curve of extremal distance between 2 points
  - ↳ on manifolds with a metric connection, geodesics are the same as auto-parallel curves, which parallel-transport their tangent vector
- The tangent vector to a curve $x^a(u)$ is $t^a \equiv \frac{dx^a}{du}$
  - ↳ curve is timelike if $g_{ab}t^a t^b > 0$, spacelike if $g_{ab}t^a t^b < 0$, null otherwise. The character of t^a can change along a curve.
  - ↳ for a non-null curve, the length of a tangent vector is ds/du

$$|t| = |g_{ab}t^a t^b|^{1/2} = \left|g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}\right|^{1/2} = ds/du$$

- The geodesic between points A and B on a manifold can be found using the Euler-Lagrange equations:
  - ↳ consider curve $x^a(u)$ parameterised s.t. $A: u=0$, $B: u=1$

$$L = \int_A^B ds = \int_0^1 \underbrace{|g_{ab}\dot{x}^a\dot{x}^b|^{1/2}}_{F = \frac{ds}{du}} du, \quad \dot{x}^a = \frac{dx^a}{du}$$

invariant to parameterisation

  - ↳ extremise with E-L: $\frac{\partial F}{\partial x^a} = \frac{d}{du}\left(\frac{\partial F}{\partial \dot{x}^a}\right)$

$$\left.\begin{aligned}\frac{\partial F}{\partial x^a} &= \pm\frac{1}{2F}\left(\partial_a g_{bc}\right)\dot{x}^b\dot{x}^c \\ \partial F/\partial \dot{x}^a &= \pm\frac{1}{F}g_{ab}\dot{x}^b\end{aligned}\right\} \begin{aligned}&+ \text{ for timelike,}\\ &- \text{ for spacelike}\end{aligned}$$

$$\Rightarrow \frac{d}{du}\left(\frac{1}{F}g_{ab}\dot{x}^b\right) = \frac{1}{2F}\partial_a g_{bc}\dot{x}^b\dot{x}^c$$

$$-\frac{1}{F^2}\frac{dF}{du}g_{ab}\dot{x}^b + \frac{1}{F}\partial_c g_{ab}\dot{x}^b\dot{x}^c + \frac{1}{F}g_{ab}\ddot{x}^b = \frac{1}{2F}\partial_a g_{bc}\dot{x}^b\dot{x}^c$$

$$\therefore\ g_{ab}\ddot{x}^b = \frac{1}{F}\frac{dF}{du}g_{ab}\dot{x}^b - \underbrace{\tfrac{1}{2}\left(2\partial_c g_{ab} - \partial_a g_{bc}\right)}_{= \frac{1}{2}(\partial_c g_{ab} + \partial_b g_{ac} - \partial_a g_{bc}) \,\equiv\, g_{ad}\Gamma^d_{bc}}\dot{x}^b\dot{x}^c$$

$$\Rightarrow \boxed{\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = \left(\frac{\ddot{s}}{\dot{s}}\right)\dot{x}^a}$$ ← Geodesic Equation (non-null geodesics).

↳ can be written explicitly as tensor equation: $\frac{Dt^a}{Du} = \left(\frac{\ddot{s}}{\dot{s}}\right)t^a$

↳ there exists an affine parameterisation such that

$u = as + b \Rightarrow \ddot{s} = 0 \Rightarrow \ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$

- In the affine parameterisation, $\frac{Dt^a}{du} = 0 \Rightarrow |\underline{t}| = \left|\frac{ds}{du}\right| = \text{const}$, so the tangent is indeed parallel-transported.
  - ↳ for null geodesics, we cannot extremise since $F=0$, so we define them as curves with parallel-transported null tangent vectors
  - ↳ the character of a tangent vector is preserved along a geodesic (since parallel-transported)
- Using the affine param, there is an alternative Lagrangian approach:

$$\frac{Dt^b}{Du} = 0 \Rightarrow \frac{Dt_a}{Du} = 0 \Rightarrow \frac{dt_a}{du} - \Gamma^c_{ba}t^b t_c = 0$$

$$\therefore \frac{dt_a}{du} = \frac{1}{2}(\partial_a g_{bc})t^b t^c$$

↳ this is consistent with applying E-L to $L = g_{ab}\frac{dx^a}{du}\frac{dx^b}{du}$, so we can just use that immediately

↳ this approach sacrifices parameterisation-invariance.

↳ e.g in $\mathbb{R}^2$, $g_{ab} = \delta_{ab} \Rightarrow L = \dot{x}^2 + \dot{y}^2$. E-L $\Rightarrow \ddot{x} = 0$

$\Rightarrow x = au + b$ (straight line)

- If the manifold has a symmetry, so g_{ab} independent of some particular x^c:
  ↳ $\partial_c g_{ab} = 0 \Rightarrow \frac{dt_c}{du} = 0 \Rightarrow t_c = \text{const}$, i.e the cth component of the tangent vector is conserved.

Minkowski Spacetime

- Minkowski spacetime is a 4D pseudo-Riemannian manifold with metric $\eta_{\mu\nu} = \text{diag}(+1,-1,-1,-1)$
  - ↳ the coordinates x^μ correspond to Cartesian coordinates in an inertial frame with $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$.
  - ↳ the metric is flat so we can choose global inertial coordinates, so the derivative of the metric vanishes $\Rightarrow \Gamma^\mu_{\nu\rho} = 0$
- Lorentz transforms (LTs) are just coordinate transforms. But with our global coordinates, the metric is unchanged: $\eta_{\mu\nu} = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu}\eta_{\rho\sigma}$
  - ↳ this implies that LTs are linear:

$$\boxed{x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \text{with} \quad \eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}}$$

  - ↳ a^μ is a constant corresponding to a shift in origin. Transform is homogeneous if $a^\mu = 0$, Poincaré otherwise.
  - ↳ for a Lorentz boost in standard config, $\boxed{\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$
  - ↳ the inverse LT is $\Lambda_\mu{}^\nu \equiv (\Lambda^{-1})^\nu{}_\mu = \eta_{\mu\rho}\eta^{\nu\sigma}\Lambda^\rho{}_\sigma$
- We only consider proper Lorentz transformations (same spatial handedness and excludes time reversal)
  - ↳ General LTs have $\underline{\eta} = \underline{\Lambda}^T\underline{\eta}\underline{\Lambda} \Rightarrow \det\underline{\Lambda} = \pm 1$
  - ↳ we require $\det\underline{\Lambda} = +1$ and $\Lambda^0{}_0 \geq 1$ for a proper LT.

- The inner product of coordinate basis vectors gives the components of the metric: $g(\underline{e}_a, \underline{e}_b) = g_{cd}(e_a)^c(e_b)^d = g_{ab}$
  - in Minkowski space, we have $\underline{e}'_\mu = \Lambda_\mu{}^\nu \underline{e}_\nu$ ← inverse transform
  - the LT does not change the metric, so $g(\underline{e}_\mu', \underline{e}_\nu') = \eta_{\mu\nu}$ and thus the new basis vectors are still orthonormal.

4-velocity

- 4-vectors' components transform via: $V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu = \Lambda^\mu{}_\nu V^\nu$
  - space/time/light-like character determined by the sign of $g(\underline{V}, \underline{V}) = ds^2$
  - a timelike/null vector is future-pointing if $V^0 > 0$, else past-pointing.
  - the metric allows us to associate vectors with duals:
    $V_\mu = \eta_{\mu\nu} V^\nu \Rightarrow V_0 = V^0, \quad V_i = -V^i$ for $i = 1,2,3$.
- For a particle with worldline $x^\mu(\tau)$, the 4-velocity is the tangent to the worldline: $u^\mu = \frac{dx^\mu}{d\tau}$
  - proper time is an affine parameter because $ds^2 = c^2 d\tau^2$
  - 4-velocity is future-pointing and timelike:
    $$g_{\mu\nu} u^\mu u^\nu = g_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \left(\frac{ds}{d\tau}\right)^2 = c^2 > 0$$
  - the 4-velocity can be written in terms of the 3-velocity
    $$u^\mu = \left(c\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right) = \frac{dt}{d\tau}\left(c, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$
    $$\boxed{u^\mu \equiv \frac{dt}{d\tau}(c, \vec{u})}$$ ← abuse of notation
  - normalisation of 4-velocity gives the relationship between coordinate and proper time. $c^2 = \eta_{\mu\nu} u^\mu u^\nu = \left(\frac{dt}{d\tau}\right)^2 (c^2 - |\vec{u}|^2)$
    $$\Rightarrow \frac{dt}{d\tau} = \left(1 - \frac{|\vec{u}|^2}{c^2}\right)^{-1/2} \equiv \gamma_u . \quad \therefore \quad u^\mu = \gamma_u \cdot (c, \vec{u})$$

- The 4-velocity can be LT'd via $u'^\mu = \Lambda^\mu{}_\nu u^\nu$
  $$\begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} \vec{u}' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\beta\gamma_v & 0 & 0 \\ -\beta\gamma_v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u \vec{u} \end{pmatrix}$$
  - 0-component gives the relationship between Lorentz factors:
    $$\gamma_{u'} = \gamma_v \gamma_u (1 - \beta \vec{u}_x / c)$$
  - i-components $(i = 1,2,3)$ give $\vec{u}'_x = \frac{\vec{u}_x - \beta c}{1 - \beta \vec{u}_x / c}$

4-acceleration

- In an inertial frame, a free particle has $\vec{\mu} = \text{const}$, $\gamma_u = \text{const}$
  $\Rightarrow \frac{du^\mu}{d\tau} = 0$
  - turn into tensor equation by using $\frac{D}{D\tau}$: metric connection vanishes in global Cartesians $\Rightarrow \frac{Du^\mu}{d\tau} = 0$
  - this implies that free massive particles move on timelike geodesics in Minkowski space
  - equivalence principle means that this holds in general curved spacetime.
- An external (non-gravitational) force will cause the particle to accelerate. Define the 4-acceleration: $a^\mu = \frac{Du^\mu}{D\tau}$
  - can use ordinary derivative in Cartesian coordinates:
    $$a^\mu = \frac{du^\mu}{d\tau} = \gamma_u \frac{d}{dt}\left[\gamma_u (c, \vec{u})\right] = \gamma_u \left(c \frac{d\gamma_u}{dt}, \frac{d\gamma_u}{dt}\vec{u} + \gamma_u \vec{a}\right)$$
  - $\boxed{\frac{d\gamma_u}{dt} = \frac{\gamma_u^3}{c^2} \vec{u} \cdot \vec{a}} \Rightarrow \boxed{a^\mu = \gamma_u^2 \left(\frac{\gamma_u^2}{c^2} \vec{u} \cdot \vec{a}, \ \vec{a} + \frac{\gamma_u^2}{c^2} (\vec{u} \cdot \vec{a}) \vec{u}\right)}$
  - 4-acceleration is orthogonal to 4-velocity
    $$0 = \frac{d}{d\tau} c^2 = \frac{d}{d\tau}(g_{\mu\nu} u^\mu u^\nu) = \frac{D}{D\tau}(g_{\mu\nu} u^\mu u^\nu) = 2 g_{\mu\nu} u^\mu \frac{Du^\nu}{D\tau} = 2 g(\underline{a}, \underline{u})$$

↳ in IRF, $\vec{u}=0 \Rightarrow a^\mu = (0, \vec{a}_{IRF})$. Hence a^μ is a spacelike vector with sq. magnitude $\eta_{\mu\nu}a^\mu a^\nu = -|\vec{a}_{IRF}|^2$

Dynamics of particles

- For a particle with rest mass m, the momentum 4-vector is defined by $p^\mu = mu^\mu$
  - ↳ $g(u,u) = c^2 \Rightarrow |p|^2 = g(p,p) = m^2c^2$ ← invariant.
  - ↳ p^μ has components $p^\mu = (\gamma_u mc, \vec{p})$ where $\boxed{\vec{p} \equiv \gamma_u m\vec{u}}$ is the relativistic 3D momentum.
  - ↳ This form of $\vec{p}$ means that momentum is conserved in all frames and $\vec{f} = \frac{d\vec{p}}{dt}$
- The 0-component of 4-momentum is the energy: $p^0 = \gamma_u mc = E/c$
  - ↳ can show that the rate of work $= \vec{u}\cdot\vec{f} = \frac{d}{dt}(\gamma_u mc^2)$
    $\Rightarrow \boxed{E = \gamma_u mc^2}$
  - ↳ we can thus write $p^\mu = (\frac{E}{c}, \vec{p})$, the magnitude of which gives the energy-momentum invariant: $\boxed{E^2 - |\vec{p}|^2c^2 = m^2c^4}$
  - ↳ for a free particle, $\frac{Dp^\mu}{D\tau} = 0$
  - ↳ for isolated particles and short-ranged interactions, $\sum_{\text{particles}} p^\mu = \text{const.}$
- The 4-momentum can be changed by the 4-force:
  $$f^\mu = \frac{Dp^\mu}{D\tau} = ma^\mu$$
  - ↳ orthogonal to 4-momentum & 4-velocity.
  - ↳ can relate to 3-force: $f^\mu = \frac{dp^\mu}{d\tau} = \gamma_u \frac{d}{dt}\left(\frac{E}{c}, \vec{p}\right) = \gamma_u\left(\frac{\vec{f}\cdot\vec{u}}{c}, \vec{f}\right)$

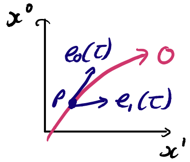
- Photons must conserve momentum. From E-p invariance, $E = |\vec{p}|c$
  - ↳ $g(p,p) = 0$ for photon so 4-momentum is a future-pointing null vector.
  - ↳ cannot use $d\tau$ as a parameter because for a null path $ds = d\tau = 0$
  - ↳ but we can choose some param λ such that $p^\mu = \frac{dx^\mu}{d\lambda}$.
    $$p^\mu = \frac{E}{c}\left(1, \frac{\vec{p}}{|\vec{p}|}\right) = \frac{E}{c^2}\left(c\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = \frac{E}{c^2}\frac{dx^\mu}{dt}$$
  - ↳ thus choose $d\lambda = c^2dt/E$, with p^μ being a tangent vector to the photon's worldline $x^\mu(\lambda)$
  - ↳ $Dp^\mu/D\lambda = 0 \Rightarrow$ free massless particles move on null geodesic in Minkowski space with an affine parameter $\lambda = \frac{c^2}{E}t$

Compton scattering

- Consider the scattering of a photon from an electron in the rest frame of the electron:

- Initially, $p^\mu = \frac{h\nu}{c}(1,1,0,0) \qquad q^\mu = mc(1,0,0,0)$
- Conserve 4-momentum: $\bar{q}^\mu = p^\mu + q^\mu - \bar{p}^\mu$
  - ↳ $|\bar{q}|^2 = |p|^2 + |q|^2 + |\bar{p}|^2 + 2g(p,q) - 2g(p,\bar{p}) - 2g(q,\bar{p})$
  - ↳ but $|\bar{q}|^2 = |q|^2 = m_e^2c^2$ and $|\bar{p}|^2 = |p|^2 = 0$
    $$\ldots \Rightarrow \bar{\nu} = \frac{\nu}{1 + (h\nu/m_ec^2)(1-\cos\theta)}$$

Local reference frame for a general observer



- For a general observer on worldline O,
  $u^\mu = \frac{dx^\mu}{d\tau}$, $a^\mu = \frac{Du^\mu}{D\tau}$
- At proper time τ, coordinate basis vectors of the IRF $\underset{\sim}{e}_\mu(\tau)$, $\mu = 0 \to 3$ form an orthonormal basis.
  - ↳ $\underset{\sim}{u}(\tau) = c\,\underset{\sim}{e}_0(\tau)$ by construction
  - ↳ $\underset{\sim}{e}_i(\tau)$, $i = 1,2,3$ span the instantaneous rest space.
  - ↳ this basis corresponds to the moving observer's local lab frame.
- The spatial vectors are only determined up to a spatial rotation. However, for a non-accelerated observer (moving on a geodesic), we can just // transport the initial frame.

Electromagnetism

Maxwell's equations in an inertial frame:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

- The Lorentz force law is $\vec{f} = q(\vec{E} + \vec{u}\times\vec{B})$
  - ↳ linear in $\vec{E},\vec{B}$, so we can try to write $f_\mu = q F_{\mu\nu} u^\nu$ where $F_{\mu\nu}$ is the type-$(0,2)$ Maxwell field strength tensor.
  - ↳ $F_{\mu\nu}$ and $F^{\mu\nu}$ are antisymmetric so that f_μ is orthogonal to 4-velocity.
  - ↳ Find components of $F_{\mu\nu}$ by matching components to $f_\mu = \gamma_u\left(\frac{\vec{f}\cdot\vec{u}}{c}, -\vec{f}\right)$
    $\therefore\ f_\mu = q F_{\mu\nu} u^\nu = q\gamma_u\left(\frac{\vec{E}\cdot\vec{u}}{c}, -\vec{E} - \vec{u}\times\vec{B}\right)$
  - ↳ $f_0 = q F_{0\nu} u^\nu = q\gamma_u \vec{E}^\nu \vec{u}^\nu / c$
    $\Rightarrow q\gamma_u F_{0\nu}\vec{u}^\nu = q\gamma_u \vec{E}^\nu \vec{u}^\nu / c \quad \therefore\ F_{0\nu} = \frac{\vec{E}^\nu}{c}$
  - ↳ $f_i = q F_{i0} u^0 + q F_{ij} u^j = -q\gamma_u[\vec{E}^i + (\vec{u}\times\vec{B})^i]$
    $\Rightarrow -q\frac{\vec{E}_i}{c} c\gamma_u + q\gamma_u F_{ij}\vec{u}^j = -q\gamma_u\vec{E}^i - q\gamma_u(\vec{u}\times\vec{B})^i$
    $\Rightarrow F_{12} = -\vec{B}^3,\ F_{13} = \vec{B}^2,\ F_{23} = -\vec{B}^1$ (cyclic sign)

$$\therefore \quad \boxed{F_{\mu\nu} = \begin{pmatrix} 0 & \vec{E}^1/c & \vec{E}^2/c & \vec{E}^3/c \\ -\vec{E}^1/c & 0 & -\vec{B}^3 & \vec{B}^2 \\ -\vec{E}^2/c & \vec{B}^3 & 0 & -\vec{B}^1 \\ -\vec{E}^3/c & -\vec{B}^2 & \vec{B}^1 & 0 \end{pmatrix}}$$

← change to $F^{\mu\nu}$ by flipping 0-i, i-0 signs

- We now know how Maxwell's eqs transform relativistically.
  - LT: $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \Rightarrow F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} F^{\rho\sigma}$

$$\vec{E}' = \begin{pmatrix} E'^{1} \\ \gamma(\vec{E}^{2} - v\vec{B}^{3}) \\ \gamma(\vec{E}^{3} + v\vec{B}^{2}) \end{pmatrix} \qquad \vec{B}' = \begin{pmatrix} \vec{B}'^{1} \\ \gamma(\vec{B}^{2} + v\vec{E}^{3}/c^{2}) \\ \gamma(\vec{B}^{3} - v\vec{E}^{2}/c^{2}) \end{pmatrix}$$

  - these are consistent with the fields resulting from length-contracted current elements and Ampere's law.

Coordinate-free Maxwell equations

- Consider a current distribution $\vec{J}$ formed by a charge density ρ moving with 3-velocity $(v,0,0)$ in S, ie $\vec{J} = \rho(v,0,0)$.
- Let S' be the rest frame of the charges (standard config) with charge density ρ_0 (no current in S')
  - length contracted in S but charge same $\Rightarrow \rho = \gamma_v \rho_0$
  - this is consistent with a current 4-vector $j^{\mu} = (c\rho, \vec{J})$
- We want to relate the field-strength tensor to the current 4-vector, linear in spacetime derivatives. We try an equation of the form $\nabla_{\mu} F^{\mu\nu} = k j^{\nu}$
  - in global inertial coords, antisymmetry of $F^{\mu\nu}$ implies charge continuity: $\partial_{\mu} F^{\mu\nu} = k j^{\nu} \Rightarrow \partial_{\nu}\partial_{\mu} F^{\mu\nu} = 0 = k\partial_{\nu} j^{\nu}$ ← divergence
  - $\nu = 0 \Rightarrow \dfrac{\partial F^{00}}{\partial(ct)} + \dfrac{\partial F^{i0}}{\partial x^{i}} = k j^{0} \Rightarrow \vec{\nabla}\cdot\vec{E} = kc^{2}\rho$

    giving one of Maxwell's equations where $k = \mu_0$
  - $\nu = i$ derives the curl terms in $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^{2}}\frac{\partial\vec{E}}{\partial t}$
- Source-free Maxwell eqs require a different tensor eq.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \quad \vec{\nabla}\cdot\vec{B} = 0 \qquad \text{(4 equations)}$$

  - we need a tensor eq that involves the covariant deriv of field strength and 4 independent components
  - consider $\nabla_{[\mu} F_{\nu\rho]} = 0$. In global inertial coordinates:

$$\partial_{\mu} F_{\nu\rho} + \partial_{\nu} F_{\rho\mu} + \partial_{\rho} F_{\mu\nu} = 0$$

  - the 4 choices of indices are $(0,1,2)$, $(0,1,3)$, $(0,2,3)$, $(1,2,3)$, each of which gives one of the equations.
- Maxwell's equations are then just the cartesian components of the tensor equations

$$\boxed{\nabla_{\mu} F^{\mu\nu} = \mu_0 j^{\nu} \qquad \nabla_{[\mu} F_{\nu\rho]} = 0}$$

  - equivalence principle implies that this holds in local inertial coordinates.

Spacetime Curvature

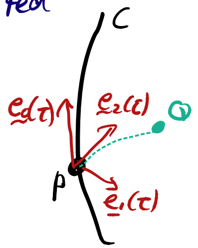
- General spacetime must be a pseudo-Riemannian manifold because (by the equivalence principle) it should reduce to Minkowski spacetime, where $ds^2 \approx \eta_{\mu\nu}\, dX^\mu dX^\nu$
- Gravity must be spacetime curvature, otherwise we could extend local inertial coordinates to all spacetime and there would be no observable effects.
- For a general manifold, we can find a free-falling non-rotating frame (i.e local inertial coordinates) such that

$$g_{\mu\nu}(P) = \eta_{\mu\nu} \quad \text{and} \quad (\partial_\rho g_{\mu\nu})|_P = 0$$

↳ however, as we move away from P, spacetime looks less like Minkowski spacetime:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{2}\left(\frac{\partial^2 g_{\mu\nu}}{\partial x^\rho \partial x^\sigma}\right)_P (X^\rho - X^\rho(P))(X^\sigma - X^\sigma(P)) + \dots$$

- We can construct Fermi-normal coordinates everywhere on the path of a free-falling observer (timelike geodesic)
  - ↳ consider an observer carrying a //-transported orthonormal frame $\{e_\mu(\tau)\}$ along their worldline.
  - ↳ any point Q near C can be connected to a point P on C by a spacelike geodesic orthogonal to e_0 at P



  - ↳ we assign 4-coordinates $X^\mu(Q) = (c\tau, s\hat{n}^i)$ where $\hat{n}^i$ are direction cosines of the geodesic at P.

Newtonian free fall

- For weak fields and slow particles, the geodesic equation reduces to Newtonian free fall.
- Weak field ⇒ global coordinates ≈ Minkowski coordinates, i.e

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad |h_{\mu\nu}| \ll 1$$

(assume metric is stationary, i.e $\partial g_{\mu\nu}/\partial x^0 = 0$)

- For slow moving particles: $\left|\frac{dx^i}{dt}\right| \lll c \Rightarrow \left|\frac{dx^i}{d\tau}\right| \ll \frac{dx^0}{d\tau}$
  - ↳ we thus don't care about $i=1,2,3$ terms in the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00} c^2 \left(\frac{dt}{d\tau}\right)^2 \approx 0$$

- Connection coefficients:

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu}(2\cancel{\partial_0 g_{\nu 0}} - \partial_\nu g_{00})$$ vanish due to stationary metric

$$= -\frac{1}{2}\sum_i g^{\mu i}\partial_i h_{00} \approx -\frac{1}{2}\sum_i \eta^{\mu i}\partial_i h_{00}$$ ← 1st order in $h_{\mu\nu}$

i.e $\Gamma^0_{00} \approx 0, \quad \Gamma^i_{00} \approx \frac{1}{2}\partial_i h_{00}$

- Geodesic equation then gives:
  - ↳ $\frac{d^2t}{d\tau^2} \approx 0 \Rightarrow \frac{dt}{d\tau} \approx \text{const}$
  - ↳ $\frac{d^2x^i}{d\tau^2} \approx -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}\left(\frac{dt}{d\tau}\right)^2 \Rightarrow \frac{d^2x^i}{dt^2} \approx -\frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i}$
- This recovers the Newtonian result $\frac{d^2x^i}{dt^2} = -\frac{\partial\Phi}{\partial x^i}$ if $g_{00} \approx 1 + \frac{2\Phi}{c^2}$
  - ↳ good approx as long as $|\Phi| \ll c^2$
  - ↳ this holds in most situations, except e.g black hole event horizons.

Intrinsic curvature of a manifold

- A manifold is flat if there are global cartesians X^a such that $ds^2 = \varepsilon_1 (dX^1)^2 + \dots + \varepsilon_N (dX^N)^2$, $\varepsilon_a = \pm 1$
- The Riemann Curvature Tensor (RCT) describes the intrinsic curvature of a manifold (independent of coordinates)
  - ↳ by construction, the covariant derivative commutes on scalar fields, i.e $\nabla_a \nabla_b \phi = \nabla_b \nabla_a \phi$. Not true on tensor fields
  - ↳ e.g for a dual-vector field:

$$\nabla_a \nabla_b V_c = \partial_a (\nabla_b V_c) - \Gamma^d_{ab} \nabla_d V_c - \Gamma^d_{ac} \nabla_b V_d$$
$$= \underbrace{\left[\partial_a \partial_b V_c - \Gamma^d_{bc} \partial_a V_d - \Gamma^d_{ac} \partial_b V_d - \Gamma^d_{ab} (\partial_d V_c - \Gamma^e_{dc} V_e)\right]}_{\text{antisymmetric}}$$
$$- (\partial_a \Gamma^d_{bc}) V_d + \Gamma^d_{ac} \Gamma^e_{bd} V_e$$
$$\Rightarrow \boxed{\nabla_a \nabla_b V_c - \nabla_b \nabla_a V_c = R_{abc}{}^d V_d}$$

  - ↳ the RCT is a type-(1,3) tensor
    $R_{abc}{}^d = -\partial_a \Gamma^d_{bc} + \partial_b \Gamma^d_{ac} + \Gamma^e_{ac} \Gamma^d_{be} - \Gamma^e_{bc} \Gamma^d_{ae}$
  - ↳ because the RCT involves derivatives of the connection, it is related to 2nd derivatives of the metric.
- Flat manifolds $\iff R_{abc}{}^d = 0$
- RCT is antisymmetric in the first two indices
  $$R_{abc}{}^d = -R_{bac}{}^d$$
  and has cyclic symmetry: $R_{abc}{}^d + R_{cab}{}^d + R_{bca}{}^d = 0$
- Further symmetries can be seen by considering $R_{abcd} = g_{de} R_{abc}{}^e$
  - ↳ can show symmetry in local cartesians (must then be true generally)
    $(R_{abcd})_p = \frac{1}{2}(\partial_a \partial_d g_{bc} + \partial_b \partial_c g_{ad} - \partial_a \partial_c g_{bd} - \partial_b \partial_d g_{ac})_p$
  - ↳ RCT is thus antisymmetric in the last two indices: $R_{abcd} = -R_{abdc}$
  - ↳ ... and is symmetric swapping 1st and 2nd pairs of indices, i.e $R_{abcd} = R_{cdab}$

- In 1D, the RCT vanishes because there is no nonzero tensor with one index that is antisymmetric (i.e $a_i = -a_i \Rightarrow a_i = 0$)
  - ↳ all lines are flat (even if embedding may be curved)
- In 2D, RCT has one component: first two indices must be diff (and there are only 2 available), while last two must = first two.
- In 3D, RCT has 6 indep. components.
  - ↳ first and last pairs have $^3C_2$ choices: 12, 13, 23
  - ↳ RCT is like a 3×3 matrix, so 6 indep. components
  - ↳ no further constraints from cyclic symmetry.
- In 4D, RCT has 20 components:
  - ↳ $^4C_2 = 6$ choices for first/last pairs
  - ↳ $6+5+4+\dots+1 = 21$ components consistent with 6×6 matrix
  - ↳ BUT cyclic symmetry introduces one constraint.

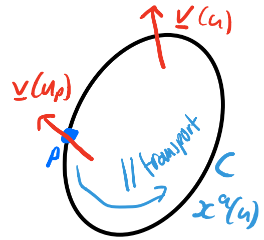
Bianchi identity and the Ricci tensor

- The Bianchi identity is a cyclic relation involving the RCT:
$$\nabla_a R_{bcd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e = 0$$
  ↳ can be shown in Cartesians ⇒ true generally
  ↳ $(\nabla_a R_{bcd}{}^e)_p = (-\partial_a\partial_b \Gamma^e_{cd} + \partial_a\partial_c\Gamma^e_{bd})_p$ ← cyclic perms add up to zero.
- Lower-rank tensors can be formed from the RCT by contraction; antisymmetry in first and last pair means we can only contract across one index from each pair
- The Ricci tensor is the contraction $R_{ab} \equiv R_{cab}{}^c$
  ↳ From RCT cyclic symmetry, the Ricci tensor is symmetric.
$$R_{ba} = R_{cab}{}^c = -R_{acb}{}^c - R_{bac}{}^c = R_{cab}{}^c = R_{ab}$$
- The Ricci tensor can be contracted to form the Ricci scalar
$$R = g^{ab}R_{ab}$$
  ↳ on a flat manifold $R_{abc}{}^d = 0$, $R_{ab} = 0$, $R = 0$
  ↳ BUT converse not always true, i.e. $R_{ab} = 0 \not\Rightarrow R_{abc}{}^d = 0$
- Contracting the Bianchi identity over b,e gives:
$$\nabla_a R_{cd} - \nabla_c R_{ad} + \nabla^b R_{cadb} = 0$$
  ↳ contract again over a,d to give the contracted Bianchi identity
$$\boxed{\nabla^a\left(R_{ab} - \tfrac{1}{2} g_{ab} R\right) = 0}$$
  ↳ hence define the symmetric and divergence-free Einstein tensor $\boxed{G_{ab} = R_{ab} - \tfrac{1}{2} g_{ab} R}$

related to cons. Energy, momentum.

Curvature and parallel transport

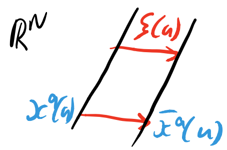
- On a manifold with intrinsic curvature, // transport is path-dependent (i.e final vector depends on path).
- Consider an infinitesimal loop C parameterised by u with a vector $\underline{v}(u)$ being //-transported.



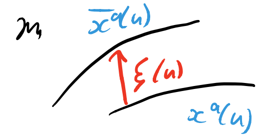
  ↳ // transport: $\frac{dv^a}{du} = -\Gamma^a_{bc}\frac{dx^b}{du}v^c$
  ↳ starting from P and // transporting gives
$$v^a(u) = v^a(u_P) - \int_{u_P}^{u} \Gamma^a_{bc}\frac{dx^b}{du'}v^c\,du'$$
  ↳ Taylor expand Γ^a_{bc} and $v^a(u)$ about P
$$\Delta v^a = v^a(u) - v^a(u_P) = -(\partial_d\Gamma^a_{bc} - \Gamma^a_{be}\Gamma^e_{dc})_P\, v^c(u_P) \oint x^d dx^b$$
  ↳ $\oint d(x^b x^d) = 0 \Rightarrow \oint x^b dx^d = -\oint x^d dx^b$ so can antisymmetrise.
$$\Rightarrow \boxed{\Delta v^a = \tfrac{1}{2} R_{bcd}{}^a|_P \; v^d|_P \oint x^{[b}dx^{c]}}$$
  ↳ i.e $\Delta \underline{v} \sim$ curvature × $\underline{v}$ × area
- Shows that $\underline{v}$ does not change on // transport if manifold is flat.

Curvature and geodesic deviation

- Curvature can cause two initially-parallel geodesics to deviate.
- Consider 2 geodesics in $\mathbb{R}^N$ with affine param u
  ↳ separation vector $\xi(u)$ is linear in u
  ↳ not true on surface of a sphere



- For 2 general geodesics $C, \bar{C}$, the connecting vector is $\xi(u) \equiv \bar{x}^a(u) - x^a(u)$.



↳ consider how ξ varies with u. Subtract geodesic equations:

$$\frac{d^2\xi^a}{du^2} + \bar{\Gamma}^a_{bc}\dot{\bar{x}}^b\dot{\bar{x}}^c - \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$$

↳ Taylor expand: $\bar{\Gamma}^a_{bc}(u) = \Gamma^a_{bc}(u) + (\partial_d\Gamma^a_{bc})\xi^d$, $\frac{d\bar{x}^b}{du} = \frac{dx^b}{du} + \frac{d\xi^b}{du}$

$$\Rightarrow \frac{d^2\xi^a}{du^2} + 2\Gamma^a_{bc}\dot{x}^b\dot{\xi}^c + \partial_d\Gamma^a_{bc}\dot{x}^b\dot{x}^c\xi^d = 0$$

↳ this can be written as the (tensor) equation of geodesic deviation:

$$\boxed{\frac{D}{Du}\left(\frac{D\xi^a}{Du}\right) - R_{dbc}{}^a\dot{x}^b\dot{x}^c\xi^d = 0}$$

- For a flat manifold, $R_{dbc}{}^a = 0 \Rightarrow \frac{D}{Du}\left(\frac{D\xi^a}{Du}\right) = \frac{d^2\xi^a}{du^2} = 0$ so ξ indeed varies linearly with u.
- In spacetime, the geodesic deviation equation describes the relative acceleration of neighbouring free-falling particles due to tidal effects.
  ↳ parameter is now τ, $u^a = \dot{x}^a$ is the 4-velocity

$$\frac{D}{D\tau}\left(\frac{D\xi^\mu}{D\tau}\right) = \underbrace{R_{\nu\alpha\beta}{}^\mu u^\alpha u^\beta}\,\xi^\nu$$

↳ $\boxed{S_{\mu\nu} = R_{\mu\alpha\beta\nu}u^\alpha u^\beta}$ is the tidal tensor (symmetric)

- In Newtonian gravity, $\frac{d^2x^i}{dt^2} = \left(-\frac{\partial\Phi}{\partial x^i}\right)_{x(t)}$, $\frac{d^2\bar{x}^i}{dt^2} = \left(-\frac{\partial\Phi}{\partial x^i}\right)_{\bar{x}(t)}$

$$\Rightarrow \frac{d^2\xi^i}{dt^2} \simeq -\left(\frac{\partial^2\Phi}{\partial x^i\partial x^j}\right)\xi^j$$

↳ in free space, $\vec{\nabla}^2\Phi = 0$ so $-\partial_i\partial_j\Phi$ is symmetric and trace-free; volume of a set of falling particles is preserved.

↳ in the weak-field slow speed limit, Newtonian = geodesic deviation

Einstein Field Equations

- Poisson's eq. does not apply generally because observers disagree on the density $\rho(\underline{r})$ (length contraction). We must generalise.
- Consider static dust (non-interacting point masses).
  ↳ in a rest frame S', the number density is n_0 so the energy density is $\rho_0 c^2 = n_0 mc^2$
  ↳ in a frame S where dust is moving uniformly at speed $\vec{u}$, num. density is $\gamma_u n_0$ due to length contraction so energy density is $\rho c^2 = (\gamma_u n_0)(\gamma_u mc^2) = \gamma_u^2\rho_0 c^2$
  ↳ energy density is not a Lorentz scalar
- The energy density transforms like the 00 component of the type-(2,0) energy-momentum tensor $\boxed{T^{\mu\nu} = \rho_0 u^\mu u^\nu}$ Lorentz scalar.
  ↳ $T^{i0} = c(\gamma_u n_0)(m\gamma_u\vec{u}^i) = c \times$ (3-momentum density)
  ↳ or $cT^{i0} = (\gamma_u^2 n_0 mc^2)\vec{u}^i$ = energy flux
  ↳ $T^{ij} = (\gamma_u^2 mn_0\vec{u}^i)\vec{u}^j$ = flux of i-component of 3-momentum in j direction.
  ↳ $T^{\mu\nu}$ is symmetric – required for cons. angular momentum.
- All sources of energy/momentum must be included in $T^{\mu\nu}$
  ↳ an ideal fluid is isotropic in its IRF, so $T^{i0} = 0$ and $T^{ij} \propto \delta^{ij}$ (for isotropy). Valid if mean free path << scale of variations.
  ↳ in the IRF $T^{\mu\nu} = \mathrm{diag}(\rho_0 c^2, p_0, p_0, p_0)$; p_0 is the pressure
  ↳ in tensor form, $T^{\mu\nu} = \left(\rho_0 + \frac{p_0}{c^2}\right)u^\mu u^\nu - p_0 g^{\mu\nu}$

↳ can find quantities e.g energy density by reading off the tensor components

↳ for a non-relativistic fluid, $\rho_0 << \rho_0 c^2$ and $T^{\mu\nu} \to \rho_0 u^\mu u^\nu$ (dust)

· Conservation of energy / momentum: $\boxed{\nabla_\mu T^{\mu\nu} = 0}$. In local inertial coordinates, consider time and space separately.

↳ $\frac{\partial T^{00}}{\partial t} + c \sum_i \frac{\partial T^{i0}}{\partial x^i} = 0$, i.e $\frac{\partial}{\partial t}(\text{energy density}) + \vec{\nabla}\cdot(\text{flux}) = 0$

↳ $\frac{\partial T^{i0}/c}{\partial t} + \sum_j \frac{\partial T^{ij}}{\partial x^j} = 0$, i.e $\frac{\partial}{\partial t}(\text{mom. density}) + \vec{\nabla}\cdot(\text{flux}) = 0$.

Field equations

· $\nabla^2 \Phi = 4\pi G \rho_0$ and the weak-field limit of the geodesic equation is $g_{00} \simeq (1 + \frac{2\Phi}{c^2})$ $\Rightarrow$ $\nabla^2 g_{00} \approx \frac{8\pi G \rho_0}{c^2} = \frac{8\pi G}{c^4} T_{00}$

· Based on this, we might look for a tensor such that

$$K_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

↳ must be symmetric and type-(0,2)

↳ should relate to spacetime curvature

↳ $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \nabla^\mu K_{\mu\nu} = 0$

↳ the Einstein tensor is a good candidate because $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ satisfies $\nabla^\mu G_{\mu\nu} = 0$

· The Einstein field equations (EFEs) are then

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}}$$

↳ set of 10 coupled PDEs for the metric functions

↳ contract with $g^{\mu\nu}$: $\boxed{R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)}$ $T \equiv g^{\mu\nu} T_{\mu\nu}$

· We can show that the EFEs recover Poisson's eq:

↳ for a non-relativistic fluid, $g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}$

↳ $T_{\mu\nu} \approx \rho_0 u_\mu u_\nu$, $T = g^{\mu\nu} \rho u_\mu u_\nu = \rho_0 c^2$

$\Rightarrow T_{00} - \frac{1}{2} g_{00} T \approx \frac{1}{2} \rho_0 c^2$

$\Rightarrow R_{00} \approx -\frac{4\pi G}{c^4} \rho_0 c^2$

↳ $R_{\nu\rho} \approx -\partial_\mu \Gamma^\mu_{\nu\rho} + \partial_\nu \Gamma^\mu_{\mu\rho}$, keeping first-order $h_{\mu\nu}$.

$\Rightarrow R_{00} \approx -\sum_i \partial_i \Gamma^i_{00}$

↳ but from the Newtonian limit of the geodesic equation, $\Gamma^i_{00} \approx \frac{1}{2} \frac{\partial h_{00}}{\partial x^i}$ with $h_{00} \approx \frac{2\Phi}{c^2}$.

$\Rightarrow R_{00} \approx -\frac{4\pi G \rho_0}{c^2} \approx -\frac{1}{2} \cdot \frac{2}{c^2} \vec{\nabla}^2 \Phi \Rightarrow \vec{\nabla}^2 \Phi = 4\pi G \rho_0$ QED.

Cosmological Constant

· We can add an additional term to the EFE:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$$

↳ Λ is the cosmological constant

↳ Lovelock's theorem shows that this is the only other tensor that satisfies the EFE in 4D spacetime.

↳ can rewrite as $R_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) + \Lambda g_{\mu\nu}$

· In the weak-field limit with small Λ, Poisson's equation becomes $\vec{\nabla}^2 \Phi = 4\pi G \rho - \Lambda c^2$

↳ for a point mass M at the origin, $F(\vec{x}) = -\frac{GM}{|\vec{x}|^3} \vec{x} + \frac{\Lambda c^2}{3} \vec{x}$

↳ linear repulsion means that universe's expansion is accelerating.

- We can think of Λ as arising from a universal negative pressure in the vacuum
  - ↳ for an ideal fluid with $p_0 = -\rho_0 c^2$, $T^{vac}_{\mu\nu} = -p_0 g_{\mu\nu} = \rho_0 c^2 g_{\mu\nu}$
  - ↳ replace $T_{\mu\nu} \to T^{vac}_{\mu\nu}$ in the EFE and compare terms: $\rho_{vac} c^2 = \Lambda c^4 / 8\pi G$
- Hypothesise that this vacuum pressure arises from the zero-point energy of quantum fields
  - ↳ considering oscillation modes in a box:
  $$\rho_{vac} c^2 = \frac{2}{(2\pi)^3} \int \frac{1}{2} \hbar \omega(k) \, d^3k$$
  - ↳ integrate up to the Planck length
  $$\rho_{vac} c^2 \sim \hbar c \, l_p^{-4} \Rightarrow \Lambda_{vac} \sim 10^{70} \, m^{-2}$$
  - ↳ vacuum catastrophe: $\Lambda \sim 10^{-52} \, m^{-2}$ experimentally so we are 120 orders of magnitude off.

Schwarzschild Solution

- Describes spacetime in a vacuum outside a spherically-symmetric non-rotating mass distribution.
- In GR, we should think of symmetries passively. Spacetime possesses a symmetry if $g'_{\mu\nu}(x')$ has the same functional form as $g_{\mu\nu}(x)$ under a coordinate transformation $x^\mu \to x'^\mu$
- Write the general line element with space and time separated (implicit sum over spatial indices $i = 1,2,3$):
  $$ds^2 = g_{00}(t,\vec{x})\,dt^2 + 2g_{0i}(t,\vec{x})\,dt\,dx^i + 2g_{ij}(t,\vec{x})\,dx^i dx^j$$
  - ↳ under a spatial rotation, $x = \underline{O} x'$ for an orthogonal matrix O. For spherical symmetry, ds^2 must have the same functional dependence on x, x':
  $$\text{①} \quad g_{00}(t,\vec{x}) = g_{00}(t, \underline{O}\vec{x})$$
  $$\text{②} \quad g_{0i}(t,\vec{x})\,dx^i = g_{0i}(t, \underline{O}\vec{x})\, O^i{}_j \, dx^j$$
  $$\text{③} \quad g_{ij}(t,\vec{x})\,dx^i dx^j = g_{ij}(t, \underline{O}\vec{x})\, O^i{}_k O^j{}_l \, dx^k dx^l$$
  - ↳ this constrains the form of the metric:
  $$\text{①} \Rightarrow g_{00}(t,\vec{x}) = A(t,r)$$
  $$\text{②} \Rightarrow g_{0i}(t,\vec{x})\,dx^i = -B(t,r)\,\vec{x}\cdot d\vec{x}$$
  $$\text{③} \Rightarrow g_{ij}(t,\vec{x})\,dx^i dx^j = -C(t,r)(\vec{x}\cdot d\vec{x})^2 - D(t,r)\, d\vec{x}\cdot d\vec{x}$$
- Require two additional symmetries:
  1. Stationary - symmetry under time translation
  2. Static - stationary and symmetric under $t \to -t \Rightarrow g_{0i} = 0$

e.g. current in infinite wire is stationary but not static

- Rewrite in spherical coordinates (redefine ABCD)

  $ds^2 = A(t,r)dt^2 - 2B(t,r)dtdr - C(t,r)dr^2 - D(t,r)d\Omega^2$

  ↳ use new radial coord $\bar r$: $\bar r^2 = D(t,r)$

  $\Rightarrow ds^2 = A(t,\bar r)dt^2 - 2B(t,\bar r)dt\,d\bar r - C(t,\bar r)d\bar r^2 - \bar r^2 d\Omega^2$

  ↳ introduce $\bar t = f(t,\bar r)$ and remove $dt\,d\bar r$ by completing the square, using an integrating factor.

- Result is a diagonal form for the isotropic line element:

  $$\boxed{ds^2 = A(t,r)\,dt^2 - B(t,r)\,dr^2 - r^2 d\Omega^2}$$

  ↳ drop t dependence if static.

  ↳ solve for $A(r)$, $B(r)$ using the EFE which reduce to $R_{\mu\nu} = 0$ in a vacuum.

  ↳ solving ODEs gives $A(r) = \alpha\left(1+\frac{k}{r}\right)$, $B(r) = \left(1+\frac{k}{r}\right)^{-1}$

  ↳ constants α, k determined by comparison with the weak-field limit $ds^2 \approx \left(1+\frac{2\Phi}{c^2}\right)d(ct)^2 + \dots$

  ↳ this gives the Schwarzschild solution:

  $$\boxed{ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2 - r^2 d\Omega^2} \quad \leftarrow \mu \equiv \frac{GM}{c^2}$$

- Solution only valid in vacuum (e.g outside the star).
- There is a coordinate singularity at $r=2\mu$, but curvature is regular.
- As $r\to\infty$, metric → Minkowski. Asymptotically flat.
- Birkhoff's theorem states that any spherically-symmetric sol. of EFE is the Schwarzchild sol., so must be static.

Geodesics in Schwarzchild spacetime

- $L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = c^2\left(1-\frac{2\mu}{r}\right)\dot t^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot r^2 - r^2\dot\theta^2 - r^2\sin^2\theta\,\dot\phi^2$

  ↳ from Euler-Lagrange, $\ddot\theta + \frac{2}{r}\dot r\dot\theta - \sin\theta\cos\theta\,\dot\phi^2 = 0$. One solution is $\theta = \pi/2$, so we can consider planar motion.

  ↳ $\partial L/\partial\dot t = \text{const} \Rightarrow \left(1-\frac{2\mu}{r}\right)\dot t = k$

  ↳ $\partial L/\partial\dot\phi = \text{const} \Rightarrow r^2\dot\phi = h$ (angular momentum conserved)

  ↳ for r, easier to use the constraint $L = \left|\frac{ds}{d\lambda}\right|^2 = \text{const}$

  $\left(1-\frac{2\mu}{r}\right)c^2\dot t^2 - \left(1-\frac{2\mu}{r}\right)^{-1}\dot r^2 - r^2\dot\phi^2 = \begin{cases} c^2 & \text{massive} \\ 0 & \text{massless}\end{cases}$

- The constant k arose from time symmetry so is related to energy

  ↳ for an observer at rest, 4-velocity is $u^\mu = A\delta_0^{\ \mu}$, where A determined by normalisation $c^2 = g_{\mu\nu}u^\mu u^\nu \Rightarrow A = \left(1-\frac{2\mu}{r}\right)^{-1/2}$

  ↳ energy of particle with 4-momentum p^μ as measured by obs with 4-velocity u^μ is $E = g(\underline{u},\underline{p}) = g_{00}Ap^0$

  ↳ for a massive particle, $E = kmc^2\left(1-\frac{2\mu}{r}\right)^{-1/2}$ so kmc^2 is the energy measured by a stationary obs as $r\to\infty$. Need $k \geq 1$

  ↳ for a massless particle, $E = kc^2\left(1-\frac{2\mu}{r}\right)^{-1/2}$. Need $k \geq 0$.

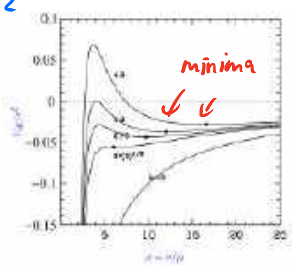
- GR thus introduces a correction to the classical orbit equation: $\frac{1}{2}\dot r^2 - \frac{GM}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2\mu}{r}\right) = \frac{1}{2}c^2(k^2-1)$

  $\dot r \equiv \frac{dr}{d\tau}$

  ↳ $V_{eff}(r) = -\frac{\mu c^2}{r} + \frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right)$

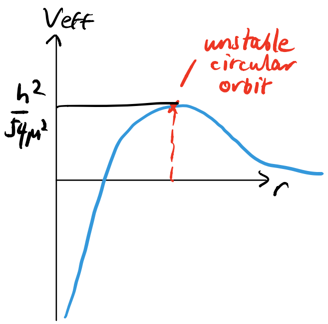
  ↳ centrifugal barrier is reversed

  ↳ very strong dependence on h



- This modified $V_{eff}(r)$ leads to different properties compared to Newtonian gravity:
  - ↳ particles can spiral in because no centrifugal barrier
  - ↳ for $h \geq \sqrt{12}\,\mu c$ there are 2 circular orbits: r_+ (stable) r_- (unstable). The innermost stable circular orbit (ISCO) is at $r = 6\mu$
  - ↳ all circular orbits for $r > 4\mu$ are bound

- For massless particles, $L=0$ so $V_{eff}(r) = \frac{h^2}{2r^2}\left(1-\frac{2\mu}{r}\right)$, i.e. no $\frac{GM}{r}$ term.
  - ↳ if $\frac{1}{2}c^2k^2 > \frac{h^2}{54\mu^2}$, incoming photons are captured
  - ↳ else a photon will deflect with pericentre given by the intersection of $E = \frac{1}{2}c^2k^2$ with $V_{eff}(r)$
  - ↳ $\frac{dr}{d\phi} = \pm r\left(\frac{c^2k^2}{h^2}r^2 - \left(1-\frac{2\mu}{r}\right)\right)^{1/2}$



compare with Newtonian to get $b = \frac{h}{ck}$, the impact parameter.

- The energy of a photon measured by a stationary observer changes along the photon path, leading to gravitational redshift.
- Photon energy is $E = g(p, u) = kc^2\left(1-\frac{2\mu}{r}\right)^{-1/2}$

$$\Rightarrow \frac{1}{1+z} = \frac{\nu_A}{\nu_B} = \frac{E_A}{E_B} = \left(\frac{1-2\mu/r_A}{1-2\mu/r_B}\right)^{1/2}$$

← $1+z \to \infty$ at $r_A = 2\mu$: event horizon

Orbits

- The shape of an orbit under the Schwarzschild metric:

$$\boxed{\frac{d^2u}{d\phi^2} + u - 3\mu u^2 = \begin{cases} GM/h^2 & (\text{massive}) \\ 0 & (\text{massless}) \end{cases}}$$

- Newtonian bound orbits described by $\frac{1}{r} = \frac{GM}{h^2}(1+e\cos\phi)$, $0 \le e < 1$, with M at the focus and $a = h^2/GM(1-e^2)$
- In the GR case, define dimensionless $U(\phi) = \frac{h^2}{GM}u(\phi)$

$$\Rightarrow \frac{d^2U}{d\phi^2} + U = 1 + \alpha U^2$$

  - ↳ $\alpha \equiv \frac{3(GM)^2}{c^2h^2} = \frac{3\mu}{r_0}$; small in the weak field limit.
  - ↳ expand $U(\phi)$ in small α: $U(\phi) = 1 + e\cos\phi + \alpha U_1(\phi) + O(\alpha^2)$
  - ↳ sub into the orbit equation to find:

$$U_1(\phi) = \left(1+\tfrac{1}{2}e^2\right) + \underline{e\phi\sin\phi} - \tfrac{1}{6}e^2\cos 2\phi$$

- Even terms first order in α are small so can be ignored, except $e\phi\sin\phi$ because this can accumulate:
  - ↳ $u(\phi) \simeq \frac{GM}{h^2}(1 + e\cos\phi + e\alpha\phi\sin\phi) \approx \frac{GM}{h^2}\left[1 + e\cos(\phi(1-\alpha))\right]$
  - ↳ hence the orbit is not closed: ϕ must increase by $\frac{2\pi}{1-\alpha}$ for r to repeat, so the orbit precesses by angle $\Delta\phi = 2\pi\left(\frac{1}{1-\alpha} - 1\right) \simeq 2\pi\alpha$ per revolution
- $\Delta\phi = 6\pi GM/a(1-e^2)c^2$; largest for small orbits with high eccentricity. GR successfully predicts the precession of Mercury.

Bending light

- The orbit of massless particles is described by $\frac{d^2u}{d\phi^2} + u = \frac{3GM}{c^2}u^2$
- In Minkowski spacetime, $M=0$ so the solution is a straight line with impact param b: $u(\phi) = \frac{1}{b}\sin\phi$
- Define dimensionless $U(\phi) = b\,u(\phi)$

$$\Rightarrow \frac{d^2U}{d\phi^2} + U = \beta U^2$$

↳ $\beta \equiv \frac{3\mu}{b}$; small in the weak-field limit.

↳ expand $U(\phi)$ in small β: $U(\phi) = \sin\phi + \beta U_1(\phi) + O(\beta^2)$

↳ sub into the orbit equation to find:

$$U_1(\phi) = C_1\sin\phi + C_2\cos\phi + \frac{1}{2}\left(1+\frac{1}{3}\cos 2\phi\right)$$

↳ B.C: $\phi \to \pi$, $U \to \sin\phi$

$$\Rightarrow u(\phi) = \frac{\sin\phi}{b} + \frac{3GM}{c^2b^2}\left(\frac{2}{3}\cos\phi + \frac{1}{2}\left(1+\frac{1}{3}\cos 2\phi\right)\right)$$

- To find the deflection angle, $u \to 0$, $\phi \to \Delta\phi$

$$\Rightarrow \boxed{\Delta\phi \approx \frac{4GM}{c^2 b}}$$

↳ this is gravitational lensing, which was experimentally verified by Eddington.

↳ double the deflection of a massive particle under Newtonian theory because light is affected by both space/time parts of the metric

Black Holes

- The Schwarzschild metric is singular when $r=0$ or $r=2\mu$

↳ $r_s = 2\mu = \frac{2GM}{c^2}$ is the Schwarzschild radius

↳ $r=0$ is a physical singularity; can be seen by considering the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \propto \frac{\mu^2}{r^6}$

↳ r_s is just a coordinate singularity; curvature is finite

- The Schwarzschild radius partitions space into two regions:

Exterior $r>r_s$	Interior $r<r_s$
$\underline{e}_0 \equiv \frac{\partial}{\partial t}$ timelike	$\underline{e}_0$ spacelike
$\underline{e}_1 \equiv \frac{\partial}{\partial r}$ spacelike	$\underline{e}_1$ timelike

↳ in the interior, a particle <u>cannot</u> stay fixed at (r,θ,ϕ) because its worldline is timelike

↳ at $r=r_s$, the time and radial coords appear to switch roles.

Causal structure near black holes

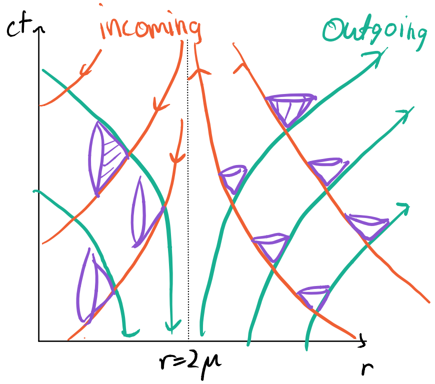
- Radial null geodesics satisfy $0 = ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt^2 - \left(1-\frac{2\mu}{r}\right)^{-1}dr^2$

$$\Rightarrow \frac{d(ct)}{dr} = \pm\left(1-\frac{2\mu}{r}\right)^{-1}$$

↳ $+ \Rightarrow ct = r + 2\mu\ln\left|\frac{r}{2\mu}-1\right| + C$ ← outgoing in interior

↳ $- \Rightarrow ct = -r - 2\mu\ln\left|\frac{r}{2\mu}-1\right| + C$ ← incoming in exterior

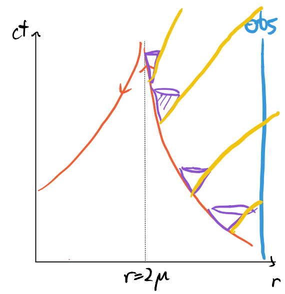
- As $r \to r_s$ from the outside, it appears to take infinite time for a photon to reach r_s
- Outgoing photons appear to originate from $r = r_s$, $t \to -\infty$



- However, these phenomena only appear in coordinate time. With an affine param λ, the Lagrangian gives $\frac{dt}{d\lambda} = k(1-\frac{2M}{r})^{-1}$
  $\Rightarrow \frac{dr}{d\lambda} = \pm kc \Rightarrow r = \pm ck\lambda + \text{const}$
  ↳ so an incoming photon does reach $r = r_s$ in finite λ
  ↳ as the photon passes through r_s, $\frac{dt}{d\lambda} < 0$ so the lightcones reorient
- The causal future of particles in the exterior includes a region in the interior where the lightcone is oriented towards $r=0$ $\Rightarrow$ unavoidable that the particle falls to the singularity. This is a BLACK HOLE
  ↳ $r = r_s$ thus defines the event horizon
  ↳ within the event horizon, particles cannot escape $r \to \infty$
- Outgoing geodesics in the exterior can be linked to an interior region in their causal past where the lightcone was oriented away from $r=0$: WHITE HOLE
  ↳ particles are expelled to $r > r_s$ from the interior
  ↳ do not exist physically → no mechanism for formation.

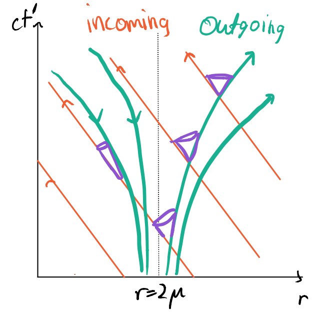
Massive infalling particles

- Setting $h=0$ for a massive particle gives $\frac{1}{2}\dot{r}^2 - \frac{GM}{r} = \frac{1}{2}c^2(k^2-1)$
  ↳ $\frac{d}{d\tau} \Rightarrow \ddot{r} = -\frac{GM}{r^2}$. Same as Newtonian, but uses proper time
  ↳ it takes the particle finite proper time to reach $r = 2\mu$ from ∞.
- We can consider the path in coordinate time:
  $\frac{d(ct)}{dr} = \frac{c\dot{t}}{\dot{r}} = -\sqrt{\frac{r}{2\mu}}\left(1-\frac{2\mu}{r}\right)^{-1}$
  $\Rightarrow c(t-t_0) = -2\mu \int_{r_0/2\mu}^{r/2\mu} \frac{x^{3/2}}{x-1}\,dx$
  ↳ diverges as $r \to 2\mu$
  ↳ a stationary observer sees the particle slow, becoming redshifted.



Eddington-Finkelstein coordinates

- We can change coordinates to avoid the $r=2\mu$ singularity for infalling particles
  $ct = -r - 2\mu \ln\left|\frac{r}{2\mu} - 1\right| + \text{const}$
  $ct' = ct + 2\mu \ln\left|\frac{r}{2\mu} - 1\right|$
  ↳ ingoing null geodesics have $ct' = -r + \text{const}$
  ↳ outgoing null geodesics are still singular
  $ct' = r + 4\mu \ln\left|\frac{r}{2\mu} - 1\right| + \text{const}$



- (t', r, θ, ϕ) are ingoing Eddington-Finkelstein (IEF) coordinates; they do not cover the causal past of the exterior (white hole)

- Use $c\,dt' = c\,dt + \left(\frac{r}{2\mu}-1\right)^{-1} dr$ to find new line element:

$$ds^2 = c^2\left(1-\frac{2\mu}{r}\right)dt'^2 - \frac{4\mu c}{r}\,dt'\,dr - \left(1+\frac{2\mu}{r}\right)dr^2 - r^2 d\Omega^2$$

↳ metric still → Minkowski as $r\to\infty$

↳ $\underline{e}_1 \equiv \frac{\partial}{\partial r}$ is now spacelike everywhere because there is no sign change at $r=2\mu$

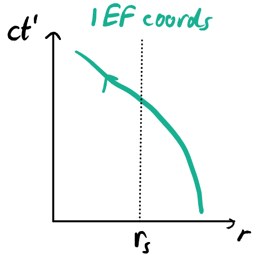
- We could instead construct outgoing EF coordinates which remove the singularity for outgoing geodesics
- IEF and OEF coordinates can be combined to form Kruskal-Szekeres coordinates which are nonsingular everywhere.

Formation of black holes

- By Birkhoff's theorem, the exterior of a star is described by the Schwarzschild solution

↳ think of the star's edge as a massive infalling particle in Schwarzschild spacetime

↳ when the edge passes $r=r_s$, the star has become a black hole.

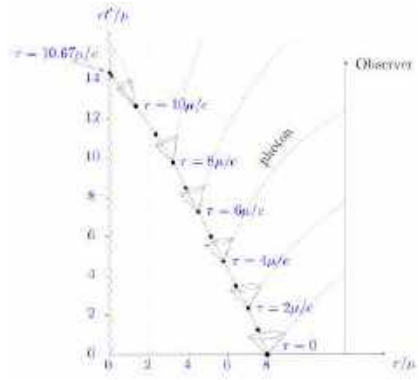


- As a simple model, we can consider the collapse of a large dust cloud from the perspective of a distant stationary observer

↳ no pressure so dust falls on geodesics.

↳ consider light rays emitted by edge of cloud at (ct'_E, r_E) and received at (ct'_R, r_R) using IEF coords

↳ we are interested in the radius of the cloud as proper time progresses



- In IEF coords, $ct'_R - r_R - 4\mu\ln\left|\frac{r_R}{2\mu}-1\right| = ct'_E - r_E - \underbrace{4\mu\ln\left|\frac{r_E}{2\mu}-1\right|}$

as $r_E\to 2\mu$, this term dominates

↳ $\tau_R \approx t'_R + \text{const}$ for a distant observer

$$\Rightarrow c\tau_R \approx ct'_R + \text{const} \approx \text{const} - 4\mu\ln\left|\frac{r_E}{2\mu}-1\right|$$

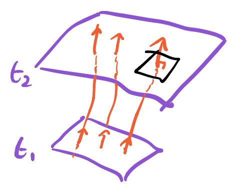
↳ the edge of the cloud is observed to approach $r=2\mu$ exponentially

$$r_E(\tau_R) \approx 2\mu\left(1+a e^{-c\tau_R/4\mu}\right)$$

↳ cloud redshifted: $\frac{\nu_R}{\nu_E} = \frac{d\tau_E}{d\tau_R} = \frac{d\tau_E}{dr_E}\frac{dr_E}{d\tau_R} \sim \left.\frac{d\tau_E}{dr_E}\right|_{r_E=2\mu} e^{-c\tau_R/4\mu}$

Cosmology

- Cosmogy aims to describe the universe as a whole using GR
- The cosmic microwave background (CMB) shows that the universe is about the same in all directions at a given time - isotropic.
- If we adopt the Copernican principle that we aren't privileged observers, then there exists a class of fundamental observers who all see the universe as isotropic and agree on what they see at a given proper time $\Rightarrow$ universe is homogeneous
- The fundamental observers co-move with matter in the universe (else there would be some v_{rel} $\Rightarrow$ not isotropic), and likewise must be free-falling else $\dot{v}_{rel}$ breaks isotropy
- The worldlines of fundamental observers must be orthogonal to hypersurfaces of constant density, else local measurements in the IRFs would reveal a spatial gradient, breaking isotropy.



- We adopt synchronous coordinates:
  - ↳ each fundamental obs has fixed spatial coordinates x^i
  - ↳ homogeneous surfaces are labelled by their proper time, which is the same for every observer – cosmic time
- The synchronous line element is $ds^2 = c^2dt^2 + g_{ij}(t,\vec{x})\,dx^i dx^j$:
  - ↳ satisfies $u^\mu \perp dx^\mu$: $u^\mu = \frac{dx^\mu}{d\tau} = \delta_0^\mu \Rightarrow g_{\mu\nu}u^\mu dx^\nu = 0 \Rightarrow g_{0i} = 0$
  - ↳ can show that it indeed satisfies the geodesic equation (timelike)

Friedmann-Robertson-Walker Metric

- The intrinsic geometry of $t = \text{const}$ hypersurfaces depends solely on spatial components of the metric $ds^2 = g_{ij}(t,\vec{x})\,x^i x^j$
- For this to be homogeneous for all t, each component of g_{ij} must evolve the same way in time, so we can factor out t-dependence

$$g_{ij}(t,\vec{x}) = -R^2(t)\,\gamma_{ij}(\vec{x})$$

  - ↳ $R(t)$ is a physical scaling factor
  - ↳ γ_{ij} is a 3D metric tensor of type-$(0,2)$
- Isotropy requires spherical symmetry, so we can use the spatial part of the Schwarzschild solution: $d\sigma^2 \equiv \gamma_{ij}\,dx^i dx^j = B(r)\,r^2 + r^2 d\Omega^2$
- We further constrain the geometry by ensuring homogeneity of the 3D metric connection, Riemann tensor, Ricci tensor, Ricci scalar.
  - ↳ Ricci scalar cannot depend on r, so set to constant ${}^{(3)}R = -6k$. But ${}^{(3)}R = -\frac{2}{r^2}\left(1 - \frac{d}{dr}\left(\frac{r}{B}\right)\right)$, so integrate to get $B = \left(\frac{A}{r} + (1-kr^2)\right)^{-1}$
  - ↳ other curvature invariants like ${}^{(3)}R_{ij}\,{}^{(3)}R^{ij}$ must also be independent of position
- The 3D line element becomes $d\sigma^2 \equiv \gamma_{ij}\,dx^i dx^j = \frac{dr^2}{(1-kr^2)} + r^2 d\Omega^2$.

  The FRW metric is then:

$$\boxed{ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right]}$$

- Homogeneous/isotropic 3D space is maximally symmetric: posseses the same number of continuous symmetries as 3D Euclidean space (3 rot + 3 translate = 6)

↳ Bianchi identity $\Rightarrow K = \text{const}$

↳ in maximally symmetric spaces, the RCT can be written as

$$R_{abcd} = K\cdot(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc})$$

↳ the 3D Ricci tensor is ${}^{(3)}R_{jk} = \gamma^{il}\,{}^{(3)}R_{ijkl} = -2k\gamma_{jk}$, so ${}^{(3)}R = -2k\gamma^{jk}\gamma_{jk} = -6k$, so this is the same k as in the FRW metric.

Intrinsic geometry of 3D spaces

- Different cases depending on the value of k.
- $k=0 \Rightarrow d\sigma^2 = dr^2 + r^2 d\Omega^2$ ← Euclidean space in polars
- For $k>0$, reparameterise $r = \frac{1}{\sqrt{k}}\sin(\sqrt{k}\chi) \equiv S_k(\chi)$

$$\Rightarrow d\sigma^2 = d\chi^2 + S_k^2(\chi)\,d\Omega^2$$

↳ this is the line element of a 3-sphere embedded in $\mathbb{R}^4$

↳ the 3D space corresponding to the surface of this 3-sphere has finite volume — a closed space

$$V = \int\sqrt{\det\gamma}\,d^3x = \int_0^{\pi/\sqrt{k}} S_k^2(\chi)\,d\chi\,d\Omega = \frac{2\pi^2}{k^{3/2}}$$

- For $k<0$, reparameterise $r = \frac{1}{\sqrt{|k|}}\sinh(\sqrt{|k|}\,\chi) \equiv S_k(\chi)$

↳ corresponds to a hyperboloid embedded in Minkowski space

↳ space is open, with infinite volume

- The FRW metric is thus $ds^2 = c^2dt^2 - R^2(t)[d\chi^2 + S_k^2(\chi)d\Omega^2]$,

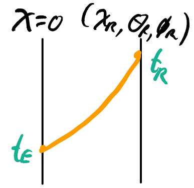
$$\boxed{S_k(\chi) = \begin{cases} \frac{1}{\sqrt{k}}\sin(\sqrt{k}\,\chi) & k>0 \quad \text{closed} \\ \chi & k=0 \quad \text{flat} \\ \frac{1}{\sqrt{|k|}}\sinh(\sqrt{k}\,\chi) & k<0 \quad \text{open} \end{cases}}$$

An expanding universe

- The proper distance between two fundamental obs at $\chi=0$, $\chi=\Delta\chi$ is $l(t) = R(t)\Delta\chi$
- The fractional rate of change in proper length is the Hubble parameter:

$$H(t) \equiv \frac{1}{R}\frac{dR}{dt} = \frac{1}{l}\frac{dl}{dt}$$

- If $H(t)>0$, the universe is expanding. In our universe, $H_0 \approx 70\,\mathrm{kms^{-1}Mpc^{-1}}$
- This expansion results in cosmological redshift:



↳ consider a radial null geodesic:

$$x^\mu(\lambda) = (t(\lambda), \chi(\lambda), \theta_R, \phi_R)$$

↳ photon 4-momentum is $p^\mu = (\dot t, \dot\chi, 0, 0)$

↳ energy measured by obs (travelling with $u^\mu = \delta^0_\mu$) is $E = p_\mu u^\mu = p_0 = c^2 p^0$

↳ the Lagrangian is $L = c^2\dot t^2 - R^2\dot\chi^2$, giving $p^1 \propto \frac{1}{R^2}$

↳ relate p^0 to p^1 using the null vector condition

$$0 = c^2(p^0)^2 - R^2(p^1)^2 \Rightarrow E = c^2p^0 \propto Rp^1 \propto \frac{1}{R}$$

↳ the redshift is thus the ratio of scale factors

$$1+z \equiv \lambda_R/\lambda_E = E_E/E_R = R(t_R)/R(t_E)$$

Cosmological field equations

$\left(\text{EFE}: R_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) + \Lambda g_{\mu\nu}\right)$

- Isotropy $\Rightarrow T^{\mu\nu}$ is that of an ideal fluid: $T^{\mu\nu} = (\rho + \frac{p}{c^2})u^\mu u^\nu - pg^{\mu\nu}$
- Homogeneity $\Rightarrow \rho, p$ only functions of t and fluid must be at rest w.r.t fundamental obs so $u^\mu = \delta^\mu_0$

- From the FRW metric we can compute the Ricci tensor/scalar:

$$ds^2 = c^2dt^2 - R^2(t)\gamma_{ij}\,dx^i dx^j$$

$$\Rightarrow \begin{cases} R_{00} = 3\ddot{R}/R \\ R_{ij} = -\frac{1}{c^2}(\ddot{R}R + 2\dot{R}^2 + 2Kc^2)\gamma_{ij} \end{cases}$$

- Compare with the result from EFEs:
  - ↳ $T = g_{\mu\nu}T^{\mu\nu} = g_{\mu\nu}\left[\left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu - p g^{\mu\nu}\right] = \rho c^2 - 3p$
  - ↳ $R_{00} \Rightarrow$ $\boxed{\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{1}{3}\Lambda c^2}$
  - ↳ $R_{ij} \Rightarrow$ $\boxed{\left(\frac{\dot{R}}{R}\right)^2 + \frac{Kc^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2}$

Friedmann equations

- Conservation of energy: $\nabla_\mu T^{\mu\nu} = 0 \Rightarrow \boxed{\dot{\rho} + 3\frac{\dot{R}}{R}\left(\rho + \frac{p}{c^2}\right) = 0}$
  - ↳ in dust, $p = 0 \Rightarrow \dot{\rho} + 3\frac{\dot{R}}{R}\rho = 0 \Rightarrow \rho \propto R^{-3}$ (intuitive)
  - ↳ if $p \neq 0$, there is PV work being done so energy falls faster, e.g. for radiation, $p = \rho c^2/3 \Rightarrow \dot{\rho} + 4\frac{\dot{R}}{R}\rho = 0 \Rightarrow \rho \propto R^{-4}$

Cosmological models

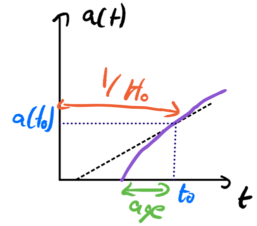
- Given the Friedmann eq. $H^2 + \frac{Kc^2}{R^2} = \frac{8\pi G}{3}\rho + \frac{1}{3}\Lambda c^2$, the evolution of the universe can be determined if we specify ρ, H, Λ, and an equation of state linking p and ρ.
- The critical density is defined by $\rho_{crit} \equiv \frac{3H^2}{8\pi G}$. If $\Lambda = 0$:

$\rho > \rho_{crit} \Rightarrow K > 0$ closed

$\rho = \rho_{crit} \Rightarrow K = 0$ flat

$\rho < \rho_{crit} \Rightarrow K < 0$ open

- If $\Lambda = 0$ and for ordinary matter with $\rho > 0$, $p \geq 0$, the 1st Friedmann equation gives $\ddot{R} < 0$.
  - ↳ in an expanding universe, this implies $R = 0$ at some finite time in the past: the Big Bang
  - ↳ the age of the universe is bounded by:
    $\text{age} < R(t_0)/\dot{R}(t_0) = 1/H_0 \approx 14\,\text{Gyr}$
  - ↳ Λ small, so $1/H_0$ is a good approx of age.



- In a flat or open universe ($K \leq 0$) with $\Lambda = 0$, $H^2 > 0$ so the universe expands forever.
  - ↳ special case of $K = 0$, $p = 0$ is an Einstein-de Sitter universe.
    $\rho \propto R^{-3} \Rightarrow H^2 \propto R^{-3} \Rightarrow R \propto \sqrt{t}$
- In a closed universe ($K > 0$) with $\Lambda = 0$, expansion eventually stops at some R_{max}, after which the universe contracts to a singularity (big crunch): $H = 0$ when $\frac{Kc^2}{R_{max}^2} = \frac{8\pi G}{3}\rho(R_{max})$
- If Λ is large enough, expansion accelerates ($\ddot{a} > 0$)
  - ↳ in the $K = 0$ case, expansion lasts forever and
    $H^2 \to \frac{1}{3}\Lambda c^2 \Rightarrow R \propto \exp\left(\sqrt{\frac{\Lambda c^2}{3}}\,t\right)$
  - ↳ this is de Sitter spacetime – maximally symmetric.