1. **Simple and Compound Interest**

**Simple Interest**

- In simple interest there is one deposit and one repayment.
  \[ C_t = C_0 (1 + r) \quad C_0 = \frac{1}{1 + r} \ C_t \]

- Over several time units, this becomes \[ C_n = C_0 (1 + nr) \].
- There are different conventions for the number of days in a year: ACT/365 assumes that there are 365 days.
- Calculations regarding simple interest must be done with respect to one fixed time point, called the focal point or valuation date.

**Compound Interest**

- Interest is given on top of existing interest. Assuming constant rate:
  \[ C_n = C_0 (1 + r)^n \]

- The discount factor is then \[ v = \frac{1}{1 + r} \].
- If an investment produces interest \[ i \], then the interest \[ i_a \] adjusted for inflation \[ i_0 \] is given by
  \[ i_a = \frac{i - i_0}{1 + i_0} \]
- By definition, the nominal rate of interest compounded \( m \)thly, \[ i^{(m)} \], leads to a growth of \( (1 + \frac{i^{(m)}}{m})^m \), in one time period.

  *e.g.* a $1000 loan for one year @ 15% p.a compounded monthly

  \[ C_n = C_0 \left( 1 + \frac{i^{(m)}}{m} \right)^m \]

  \[ C_1 = C_0 \left( 1 + \frac{i^{(m)}}{m} \right)^m \]

  \[ = 1000 \left( 1 + \frac{0.15}{12} \right)^{12} \]
The effective interest is the effective growth in one time period (annual) by default.
\[ i = \left( 1 + \frac{r}{m} \right)^m - 1 \]

- Alternatively, if the period \( p \) is given (\( p = \frac{1}{m} \)), then \( i_{eq} = \frac{p}{m} i_{cm} = \frac{r}{m} \) is called the effective interest rate over period \( p \).
- Effective rates for different time periods can be converted with:
\[ (1 + i_{eq})^q = (1 + i_{eq})^{\frac{q}{p}} \]

- Nominal rate (period \( p \)) = Frequency \times effective rate (Freq = \( \frac{1}{p} \)).
- Always assume given interest rates are compounded annually, unless the time period is less than one year, in which case you should assume simple interest.
- The accumulation factor is the accumulated value after one time period, i.e., \( A(p) = (1 + i)^p = e^{\delta} 1 + \rho \).

- In continuous compounding, we let \( m \to \infty \):
\[ i = \lim_{m \to \infty} \left( 1 + \frac{i_{eq}}{m} \right)^m = e^\delta -1 \]
- \( \delta \) is the force of interest, i.e., \( \lim_{m \to \infty} i^{(m)} \)
- \( c_n = c_0 e^{\delta} \)

The Discount rate

- The interest rate on a deposit can also be called the discount rate on the repayment, i.e., \( PV \times \text{interest rate} = FV \times \text{discount rate} \).
- The discount rate is linked to the discount factor by \( v = 1 - d \).
- For a simple discount rate \( c_0 = (1 - nd)c_n \) by definition. Thus:
\[ (1 + ni)(1 - nd) = 1 \]
Investment terminology:

- the coupon is an interest payment based on the face/nominal value of some instrument.
- The yield is the interest rate that can be earned, and is implied by the market price relative to the maturity price.
- A discount instrument does not pay a coupon; simple interest is used for calculations because there is only one repayment.

As with interest, one can have nominal or effective discount rates:

\[ 1 - d = \left( 1 - \frac{d^{(m)}}{m} \right)^m \Rightarrow \left( 1 + \frac{i^{(m)}}{m} \right) \left( 1 - d^{(m)} \right) = 1 \]

If we borrow $1 at \( t=0 \) for repayment at \( t=1 \) with annual interest \( i \):

- Repaying with equal instalments at the end of each \( m^{th} \) subinterval leads to a total interest payment of \( i^{(m)} \).
- Repaying with equal instalments at the start of each \( m^{th} \) subinterval leads to a total interest payment of \( d^{(m)} \).

For continuous compounding, \( d_{cp} = 1 - e^{-\rho_{cp}} \).

Time dependent interest rates

Suppose we have a case of continuous compounding where interest varies with time, i.e. \( i_{p}(t) \) gives the nominal interest rate for period \( p \) starting at time \( t \). Then:

\[ A(t+p) = A(t) \left[ 1 + i_{p}(t) \right] \]

In the limit as \( p \to 0 \) of \( i_{p}(t) \), we find that:

\[ g(t) = \lim_{p \to 0} \frac{A(t+p) - A(t)}{p} = \frac{A'(t)}{A(t)} \]

Thus:

\[ A(t_2) = A(t_1) \exp\left( \int_{t_1}^{t_2} g(s) ds \right) \]

The present value corresponding to the force of interest function is given by:

\[ V(t) = \frac{1}{A(t)} = \exp\left( -\int_{0}^{t} g(s) ds \right) \]
2. Cash flows, equations of value, and project appraisal

Cash flows

- Zero-coupon bonds provide a specified cash amount at a future time

- Fixed interest security: coupon is paid at \( t = 1, 2, 3, \ldots, n \), and a lump sum is paid at \( t = n \)

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-C_0)</td>
</tr>
<tr>
<td>1</td>
<td>(C)</td>
</tr>
<tr>
<td>(n-1)</td>
<td>(C)</td>
</tr>
<tr>
<td>(n)</td>
<td>(C)</td>
</tr>
</tbody>
</table>

- Indexed-link securities have coupons linked to some index

- Annuity: investor pays a premium at time 0 and receives annual payments

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
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<td>(n-1)</td>
<td>(C)</td>
</tr>
<tr>
<td>(n)</td>
<td>(C)</td>
</tr>
</tbody>
</table>

- Equity: after purchase, you will receive a dividend at constant intervals

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-C_0)</td>
</tr>
<tr>
<td>1</td>
<td>(C)</td>
</tr>
<tr>
<td>2</td>
<td>(C)</td>
</tr>
<tr>
<td>3</td>
<td>(C)</td>
</tr>
<tr>
<td>(\ldots)</td>
<td>(\ldots)</td>
</tr>
</tbody>
</table>

- Term assurance: premium paid at time 0. If death occurs before \( t = n \), the beneficiary will receive an amount \( C \).

- Loans and mortgages: borrower receives \( C_0 \) at \( t = 0 \), then repays it (capital and interest) in a series of payments

- Insurance: company receives a premium, with the amount and frequency of negative cash flows being uncertain.

Net present value and discounted cash flow

- The DCF formula states that the NPV of a series of cash flows is given by:

\[
NPV(C) = C_0 + \frac{C_1}{1 + i} + \frac{C_2}{(1 + i)^2} + \ldots + \frac{C_n}{(1 + i)^n} = \sum_{j=0}^{n} \frac{C_j}{(1 + i)^j}
\]

- Properties of NPV:

- Decreasing in \( i \), and decreases faster when later payments are larger
- If \( a_j \geq b_j \), \( j = 0, \ldots, n \), then \( NPV(a) \geq NPV(b) \) for \( i \in [0, \omega) \)
If \( \nu(t) \) is the present value of a unit amount at time \( t \),

\[
NPV(\nu) = \sum_{j=1}^{n} c_j \nu(t_j)
\]

To find the NPV for continuous cash flows \( p(t) \), we let \( c_t = p(t) \, dt \) and take the limit of the above.

\[
NPV(p) = \int_0^t p(u) \nu(u) \, du
\]

In the special case of constant interest \( i \), we have:

\[
NPV(p) = \int_0^t \frac{p(u)}{(1+i)^u} \, du
\]

For a general force of interest function:

\[
NPV(p) = \int_0^t p(u) \exp(-\int_0^u \delta(s) \, ds) \, du
\]

The accumulated value is just \( NPV \exp(\int_0^t \delta(s) \, ds) \), i.e.

\[
A(t) = \int_0^t p(u) \exp(\int_u^t \delta(s) \, ds) \, du
\]

The equation of value and IRR: comparing projects.

The equation of value / yield equation of \( i \) is generally given by \( f(i) = 0 \), where \( f(i) = NPV(\nu, p) \).

If \( f(i) = 0 \) has exactly one root \( i_o \), with \( i_o > -1 \), then the internal rate of return (IRR), or the money weighted rate of return, is defined to be \( i_o \). The IRR is the interest that zeroes the NPV.

If \( i > IRR \), NPV < 0.

Numerical methods for finding the IRR:
- expand \( 1/(1+ri) \) as \( 1-ri \)
- multiply out \( (1+ri)^n \), then expand \( (1+ri) \) as \( 1+ri + \frac{(ri)^2}{2} \)
- assume all future cash-flows equal the average cash flow

\[
\begin{array}{c|cccccc}
 t & 0 & 1 & 2 & \ldots & n-1 & n \\
 C & c_0 & c_1 & c_2 & \ldots & c_{n-1} & c_n \\
\end{array}
\]

with \( r = \sum_{j=1}^{n} c_j \).
Then, we have \( c_0 = \frac{x}{i} \left[ \frac{1 - \left( \frac{1}{1+i} \right)^n}{(1+i)} \right] = x \alpha_{n,r} \), where \( \alpha_{n,r} = \frac{1}{r} \left[ 1 - \left( \frac{1}{1+r} \right)^n \right] \).

The values for \( \alpha_{n,r} \) for different \( r \) can be found in tables.

- Distributing capital gains: if a cash flow consists of an interest payment \( x \) in years \( 0 \rightarrow n-1 \) and a final payment of \( c_n - x \), we can approximate \( i = \frac{x + (c_n - x - c_0)}{c_0} \) at the internal rate.

- Disadvantages of IRR:
  - there may be zero solutions or multiple solutions
  - the IRR assumes that accumulated funds / loans will grow

Projects may have different profitability depending on interest rates. It may be useful to graph \( NPV_i(a) \) and \( NPV_i(b) \) as functions of \( i \). The rate that gives equal NPVs is called the cross-over rate.

- The discounted payback period (DPP) is the smallest time such that the accumulated value of the project becomes positive
  - especially useful if capital is scarce
  - does not quantify profit: projects with late but large cash inflows may have higher DPP but also higher NPV.

- The payback period is the time when net cash flow is positive — it is inferior because it doesn’t discount future cash flows

Overall, NPV is the best metric when \( i \) is known. Sensitivity analyses should be performed to observe how NPV varies with \( i \).
Measuring investment performance

Many projects, e.g. dividend-paying funds, will have cash flows of the form:

<table>
<thead>
<tr>
<th>Time, $t$</th>
<th>$t_0$</th>
<th>$t_1$</th>
<th>$t_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_0$</td>
<td>$f_{t_0}$</td>
<td>$f_{t_1}$</td>
<td>$f_{t_n}$</td>
</tr>
<tr>
<td>$C_0$</td>
<td>$C_1$</td>
<td>$C_n$</td>
<td></td>
</tr>
</tbody>
</table>

The internal rate of return solves the equation of value, which equates all inflows with the eventual accumulation:

$$f_0(l+i)^t + C_1(l+i)^{t-1} + \ldots + C_n(l+i)^{-n} = f_n$$

It is also called the money-weighted rate of return (MWRR) because it is influenced more by the performance when the fund is larger.

The MWRR has some drawbacks:
- sensitive to the amount and timing of cash flows, which are not controlled by fund managers
- may not have a unique solution

Alternatively, the time-weighted rate of return (TWRR) ignores the size of the fund and instead ‘averages’ the growth between cashflows

$$\frac{(1+i)^t}{f_0} = \frac{f_{t_0}}{f_0} \frac{f_{t_1}}{f_1} \ldots \frac{f_{t_n}}{f_{t_{n-1}}}$$

However, the TWRR requires knowledge of all cash flows, and the value of the fund on the cashflow date.

The linked internal rate of return (LIRR) can be used to approximate the TWRR: it calculates the IRR at specified sub-intervals then combines them with the TWRR formula:

$$(1+i)^t = (1+i_1)^{t_1} (1+i_2)^{t_2} \ldots (1+i_n)^{t_n}$$

- the LIRR can be used if the fund is not valued at every cashflow
- LIRR = TWRR if the interval lengths are the same
3. Perpetuities, Annuities and Loans

Perpetuities

- A perpetuity is a special case of an annuity that pays a fixed amount every year forever.
  - time: 0 1 2 3 ...
  - cash flow: 0 1 1 1 ...
  - The present value at time zero is denoted as $a_{\infty}$:
    \[ a_{\infty} = \frac{1}{i} \]
  - The present value at time 1 is $\ddot{a}_{\infty}$. This quantity is equal to the value of the perpetuity with payments made in advance.
    - Clearly $\ddot{a}_{\infty} = (1+i) a_{\infty}$

Annuities

- The most common form has constant annual payments.
  - time: 0 1 2 ... n-1 n
  - cash flow: 0 1 1 ... 1 1
  - $a_n$ is the value at time zero, called the present value of the annuity-immediate $a_n$. This corresponds to payments in arrears.
    \[ a_n = \frac{1-\nu^n}{1-\nu} = \frac{1-\nu^n}{i} \]
  - $\ddot{a}_n$ is the value at time 1, the present value of the annuity-due, which corresponds to payments in advance.
    \[ \ddot{a}_n = (1+i) a_n = \frac{1-\nu^n}{1-\nu} \]
  - $s_n$ is the accumulated value of the annuity-immediate.
    \[ s_n = (1+i)^n a_n \]
A deferred annuity has its first payment at time \( k+1 \). Its present value is the same as that of a standard annuity, just scaled down by \( v^k \), i.e. \( k! a_{n}^{(m)} = v^k a_{n}^{(m)} \).

**Annuities payable monthly**

\[
\begin{array}{c|cccccccc}
\text{time} & 0 & 1/m & 2/m & \ldots & (nm-1)/m & nm/m & (nm+1)/m \\
\hline
C & 0 & 1/m & 1/m & \ldots & 1/m & 1/m & 1/m \\
\end{array}
\]

\[
a_{n}^{(m)} = \frac{1}{m} \sum_{j=1}^{nm} v^{j/m} = \frac{1-v^{n}}{c^{(m)}} = \frac{i}{c^{(m)}} a_{n}^{(m)}
\]

and \( \ddot{a}_{n}^{(m)} = (1+i)^{1/m} a_{n}^{(m)} \)

**Increasing annuities**

\[
\begin{array}{r|cccccccc}
\text{time} & 0 & 1 & 2 & \ldots & n \\
\hline
C & 0 & 1 & 2 & \ldots & n \\
\end{array}
\]

\[
(Ia)_{n} = \frac{1}{i} \left[ \frac{1-v^{n}}{1-v} - nv^n \right] = \frac{a_{n}^{(m)} - nv^{n+1}}{1-v}
\]

General results about increasing annuities can be found by considering equivalent cash flows.

\[
e.g. \begin{array}{r|cccccccc}
\text{time} & 0 & 1 & 2 & 3 & \ldots & n \\
\hline
C & 0 & p & p+q & p+2q & p+(n-1)q \\
\end{array} = \begin{array}{c}
C_1 \\
\end{array} \begin{array}{c}
0 \quad p-q \quad p-q \quad p-q \quad \ldots \quad p-q \\
+ C_2 \\
\hline
\end{array} \begin{array}{c}
x \quad 2x \quad 3x \quad \ldots \quad nx \\
\end{array}
\]

\[
\therefore \quad NPV = (p-q) a_{n}^{(m)} + Q (Ia)_{n}
\]
Continuously payable annuities

\[ \bar{a}_{n|} = \int_0^n p(t) \nu(t) dt = \int_0^n \nu v dt = \frac{1 - (1 - \nu)^n}{\delta} = \frac{i}{\delta} a_{n|} \]

Many of the previous results apply:
- \[ \frac{\bar{s}}{\bar{a}} = (1+i)^n \bar{a}_{n|} \]
- \[ \frac{\bar{s}}{\bar{a}} = \nu k \bar{a}_{n|} \]
- \[ \frac{\bar{s}}{\bar{a}} = \frac{i}{\delta} a_{n|} \]

For a continuously payable annuity increasing in steps:

\[ (1\bar{a})_{n|} = \sum_{j=1}^n jv \int_{j-1}^j \nu dt = \frac{1}{\delta} \bar{a}_{n|} - n\nu^n \]

If payment increases continuously (i.e. \( p(t) = i \))

\[ (1\bar{a})_{n|} = \int_0^n t v dt = \frac{\bar{a}_{n|}}{\delta} - n\nu^n \]

Loan schedules

Loan schedules are represented in tables:

<table>
<thead>
<tr>
<th>year</th>
<th>loan outstanding before repayment</th>
<th>repayment</th>
<th>interest due</th>
<th>capital repair</th>
<th>loan outstanding after repayment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( l_0 )</td>
<td>( x_1 )</td>
<td>( il_0 )</td>
<td>( x_1 - il_0 )</td>
<td>( l_1 = l_0 - (x_1 - il_0) )</td>
</tr>
<tr>
<td>2</td>
<td>( l_1 )</td>
<td>( x_2 )</td>
<td>( il_1 )</td>
<td>( x_2 - il_1 )</td>
<td>( l_2 = l_1 - (x_2 - il_1) )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>k</td>
<td>( l_{k-1} )</td>
<td>( x_k )</td>
<td>( il_{k-1} )</td>
<td>( x_k - il_{k-1} )</td>
<td>( l_k = l_{k-1} - (x_k - il_{k-1}) )</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>n</td>
<td>( l_{n-1} )</td>
<td>( x_n )</td>
<td>( il_{n-1} )</td>
<td>( x_n - il_{n-1} )</td>
<td>( l_n = 0 )</td>
</tr>
</tbody>
</table>

The equation of value for a loan is:

\[ l_0 = x_1 \nu + x_2 \nu^2 + \ldots + x_n \nu^n \]
There are two methods for finding $L_k$, the outstanding loan value immediately after the $k$th payment:

1. The retrospective loan calculation:
   
   \[ L_k = L_0 (1+i)^k - \frac{x_1 (1+i)^{k-1} + x_2 (1+i)^{k-2} + \ldots + x_k}{1+i} \]

2. The prospective loan calculation (generally easier):
   
   \[ L_k = L_0 \sum \frac{x}{(1+i)^n} \]

In the special case of level payments (equal instalments), the equation of value is

- $L_0 = x a_\frac{i}{(1+i)}$
- The prospective calculation shows that $L_k = x a_\frac{i}{(1+i)}$ and a capital repayment of $x n^{-\frac{k}{1+i}}$.

For the more general case of mutually payments of $\frac{1}{i} m x$, we have $L_0 = m x a^{\frac{cm}{n}}$

**Flat rate and APR**

- Banks sometimes quote a flat rate of interest, which means that interest is charged on the full amount of the loan, regardless of capital repayments.

  \[ \text{Total interest paid} = \text{Flat rate} \times \text{Size of loan} \times \text{Length of loan} \]

- The effective rate tends to be just under double the flat rate.
- The annual percentage rate (APR) is the effective rate rounded to the lower 0.1%.
4. Basic Financial Instruments

Markets, interest rates and instruments

- A primary market is where a security is originally issued.
- If the security is negotiable, it can subsequently be traded in a secondary market.
- The capital market refers to all long-term financial instruments.
- A security is a bearer security if payment is made to whoever is holding it. Registered securities include a central record of ownership.
- The money market is the market in short-term (<1 year) financial instruments which are based on an interest rate.
  - Treasury bill / T-bill: borrowing by the government, negotiable and bearer security. Carries a coupon. Called a Euro bill in Europe
  - Time deposit: a non-negotiable borrowing by a bank
  - Certificate of deposit (CD): negotiable bearer borrowing by a bank, usually carrying a coupon
  - Commercial paper (CP): negotiable bearer borrowing by company
  - Repurchase agreement (repo): non-negotiable security to borrow against a long term instrument.
  - Futures contract: a deal to buy or sell an instrument at a future date.

Functions of the money market:
- banks can lend to other banks with money shortages
- companies may have surplus/shortage of money
- The Bank of England can lend money to eligible commercial banks by buying their financial instruments, at the official dealing rate.
- Banks in the UK will lend to each other at the LIBOR.
Fixed government borrowings

- A borrower who issues a bond agrees to pay interest at a specified rate (coupon payments) until the maturity/redeemption date, at which time a fixed redemption value is paid.
- The coupon rate is applied to the face/par value of the bond, which is the value that should be paid at maturity.
  - if face value = redemption value, the bond is redeemable at par
  - if face value < redemption, the bond is redeemable at a premium
  - if face value > redemption, the bond is redeemable at a discount
- Some bonds have variable redemption dates - the borrower decides.
- Some government bonds have coupon and redemption linked to an inflation index.
- Investment banks can decompose a government bond into strips (separately traded and registered interest and principal security).
- A UK government bond is called a gilt.

Corporate / bank fixed borrowings

- Debentures (corporate bonds) are issued by companies and are usually secured against specific assets. Because they are more risky, investors will expect a higher yield.
- Foreign bonds are issued in the local currency but by a foreign borrower, e.g.:
  - Samurai bonds in Japan (yen)
  - Yankee bonds in the US
  - Bulldog bonds in the UK (£)
- Eurobonds are issued in a different currency in a different country, e.g. if you need USD in the Swiss financial market.
- Eurobonds were developed to make eurocurrency (currency deposited outside of its home country) lending more marketable.
Certificates of deposit (CDs) are issued by banks in return for a deposit. There is an active secondary market, but this flexibility reduces yield.

- because they are short term, simple interest is typically used
- the maturity proceeds are calculated by simple accumulation or the face value with coupon rate over d days

\[ M_p = f \left( 1 + \frac{ic \cdot d}{360} \right) dt \]

- if the CD is bought at time \( a \), for price \( P_a \), and subsequently sold at time \( b \), for price \( P_b \), the yield is

\[ \left( \frac{P_b}{P_a} - 1 \right) \frac{d_t}{b - a} \]

can be rearranged in terms of the yields at time \( a \) and \( b \).

Investments with uncertain returns

- Ordinary shares entitle holders to a share in the net profits of a company, the dividend.
  - they are the last to be repaid during bankruptcy, so are the most risky
  - if a share is bought ex-dividend, the seller receives the next dividend payment.

- Preference shares normally pay a fixed dividend, and have a higher debt priority than ordinary shares.

- Convertibles are loans that can be converted into ordinary shares at a fixed price and a fixed date
  - investors accept a lower interest payment because of possible capital gains from the rise in share prices
  - companies don’t have to immediately dilute earnings and dividends as would be the case for issuing shares.

- Properties provide uncertain rental incomes and volatile capital gains.

- The running yield of an investment is the annual income divided by the current market price (ignoring capital gains). Thus: bonds > property > equity.
Derivatives

- Derivatives are instruments whose value depends on other financial assets. They can be used to hedge positions or for speculation.
- Market risk is the risk that market conditions change adversely, while credit risk is the risk that the counterparty in an agreement will default on its payments.

- A forward contract is a legally binding contract to buy/sell an agreed quantity of an asset at a specified time in the future.
- A futures contract has the same definition; the difference is that forward contracts are over-the-counter (OTC), while futures are traded on an exchange and are thus more standardized.
- At maturity, the short side (seller) delivers the asset to the long side (buyer) for the exchange delivery settlement price (£50p).
- Often, a cash settlement is made instead of actually transferring assets.
- Each party in a futures contract must deposit some money, called the margin, to the clearing house to control credit risk.

- A financial future is a contract based on a financial asset.
- A bond future involves a notional bond - the exact list of eligible bonds must be agreed on. At maturity, the short side will deliver the cheapest bond.
- Stock index futures can be used to speculate on the movement of the market as a whole.
- Currency futures require the delivery of a specified amount of a given currency on a specified date.
- A forward rate agreement (FRA) is an interest rate hedge in which the buyer (normally a borrower) fixes the interest rate on his loan, while the seller receives the actual floating rate. Thus a buyer of an FRA profits if interest rates rise.
An interest rate future is a futures-contract equivalent of an FRA. The price of an interest rate future rises as interest rates fall: they are usually priced at 100 - i.

A currency swap is an agreement to exchange interest payments and a capital sum of one currency for those in another currency. This is useful because some parties may be able to get favourable rates in their home country.

In an interest rate swap, two parties swap a series of interest payments. Usually, one party will want a fixed instead of floating rate.
- with a cap, the bank agrees to pay any interest above a certain rate
- with a floor, the bank agrees to always pay you at least a certain amount of interest
- a combination of cap and floor is called a collar.

Options

- A call option gives the holder the right to to buy an asset for a specified price (the strike price) at a specified time in the future.
- A put option lets you sell at the strike price.
- An American option can be exercised at any date before expiry, while a European option can only be exercised on the expiry date.
- An option is in-the-money if immediate execution would be profitable, and out-of-the-money otherwise.

The maximum loss for the holder of an option is the premium paid.
- For a call, the holder breaks even if the price of the underlying at expiry equals the exercise price plus the premium paid.
- For a put, the holder breaks even if the price of the underlying at expiry equals the exercise price minus the premium paid.
5. **Bonds, equities and inflation**

**Bonds**

- A borrower who issues a bond pays a coupon on the face value or the bond at specified intervals until maturity, at which time they pay a fixed redemption value (almost always the same as the face value).

  **Fixed by the terms:**
  - \( f = \text{face/par value} \)
  - \( P = \text{current price} \)
  - \( r = \text{coupon rate per year} \)
  - \( i = \text{yield to maturity} \) (same as IRR)
  - \( C = \text{redemption value} \)
  - \( n = \text{time to redemption} \)

- By writing down the cashflows of the bond, it can be seen that:

\[
p = f \cdot \frac{a^{(m)}_{n, i}}{i} + C \cdot v^n
\]

  - this is the equation of value
  - price varies inversely as yield, given fixed \( n \).
  - if \( P = f = C \), then \( i^{(m)} = r \).

- If income (coupon payments) are taxed at rate \( t \):

\[
p = f \cdot r \cdot (1-t) \cdot a^{(m)}_{n, i} + C \cdot v^n
\]

- Ratio of coupon to current price has many names: gross interest yield, flat yield, direct yield, current yield, gross running yield. These exclude tax. Net interest yield or net running yield includes tax.

- The yield to maturity, also called the gross/net yield to redemption, includes all subsequent coupons as well as the final redemption.

- To calculate yield to maturity, we can approximate it as being equal to \((\text{coupon per year} + \text{capital gains per year})/\text{price}\), i.e.:

\[
i \approx \frac{1}{P} \left( fr + \frac{100-P}{n} \right)
\]

- overestimates \( i \).
For simple cases, if \( i > r \) then the bond is trading at a discount, and if \( i < r \) then the bond is trading at a premium.

To generalise slightly, let \( q = r/c \) (reduces back to the coupon rate if bond is redeemed at par). Then:

\[
\begin{align*}
\text{if } & i^{(m)} < (1-t_1)q, \text{ it means that } P > C \quad (\text{premium}) \\
\text{if } & i^{(m)} > (1-t_1)q \Rightarrow P < C \quad (\text{discount}).
\end{align*}
\]

In the latter case, capital gains tax \( (t_2) \) applies:

\[
P = \frac{C + \sum_{n=1}^{m} \frac{c \cdot a^{(m)}_{n}}{1 - t_2 v^n}}{1 - t_2 v^n}
\]

Some bonds have their redemption date variable (at the bond issuer's option). The investor cannot know the exact yield, and must assume the worst case:

- if \( i^{(m)} < (1-t_1)q \), there is no capital gain and should thus expect earliest possible redemption
- if \( i^{(m)} > (1-t_1)q \), assume latest redemption as issuer would like to defer paying out.

If a bond is sold between coupon dates, the seller feels entitled to some accrued interest. He thus sells the bond for the dirty price:

\[
\text{dirty price} = \text{NPV of future cash flows} = \text{clean price} = \text{dirty price} - \text{accrued interest}
\]

Under ACT/365:

\[
\text{accrued interest} = \text{annual coupon} \times \frac{\text{days since last coupon}}{365}
\]

Bonds quoted \textit{earn dividend mean that the next coupon will go to the buyer}.

\textit{Ex dividend means that the buyer will not receive the coupon, so accrued interest is negative.}
Equities calculations and interest with inflation

- Equities are characterised by a stream (often assumed to be infinite) of uncertain dividend payments.
- The money rate of interest of an investment ignores interest; the effective rate of interest includes the inflation adjustment.
  \[ 1 + i_R = \frac{1 + i_m}{1 + q}, \] where \( q \) is the inflation rate.

- For a general cash flow:

<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>cash flow</td>
<td>-( p )</td>
<td>( c_1 )</td>
<td>( c_2 )</td>
<td>...</td>
<td>( c_n )</td>
</tr>
<tr>
<td>inflation</td>
<td>( q(0) )</td>
<td>( q(1) )</td>
<td>( q(2) )</td>
<td>...</td>
<td>( q(n) )</td>
</tr>
</tbody>
</table>

The equation of value is given by:

\[ P = \sum_{k=1}^{n} \frac{C_k}{Q(k)} \nu^k \] with \( \nu = \frac{1}{1+i} \)

- For the special case of constant inflation

\[ P = \sum_{k=1}^{n} C_k \nu_M^k = \sum_{k=1}^{n} \frac{C_k}{(1+q)^k} \nu_R^k \] using \( \nu_R = (1+q)\nu_M \)

- Some securities have their payments/coupons linked to an inflation index. In theory, this means that \( C'_k = C_k \frac{Q(k)}{Q(0)} \), but in practice there may be a lag time.
6. Interest Rate Problems

Spot rates, forward rates and the yield curve

- The term structure of interest rates is the relationship between interest rates and maturities.
- Instead of a growth \((1+r)^t\), we have \(r\) as a function of \(t\), so the accumulation will be \((1 + y_t)^t\).
- \(y_t\) is called the \(t\)-year spot rate of interest.
- Any fixed interest security can be analysed in terms of spot rates by looking at it as a sum of zero-coupon bonds.
- For example, a bond with coupon \(c\), annually and redemption \(C\) obeys:
  \[
  P = c \left[ \frac{1}{(1+y_1)} + \frac{1}{(1+y_2)^2} + \cdots + \frac{1}{(1+y_n)^n} \right] + \frac{C}{(1+y_n)^n}
  \]
  \(y_1\) can be determined from the yield of 1y T Bill, then \(y_2\) from the price of 2y bonds, and so on:
  \[
  P = \frac{cr}{(1+y_1)} + \frac{F_r + C}{(1+y_2)^2}
  \]
  \(y_1\), known, so \(y_2\) can be found.
- A zero coupon bond can also be constructed by subtracting bonds with different coupons.
- The \(n\)-year par yield of a bond is the coupon rate that causes the bond price to be equal to its face value, assuming redemption at par. If \(r\) is the par yield, \(r\) satisfies:
  \[
  r \sum_{n=1}^{N} \frac{1}{(1+y_n)^k} + \frac{1}{(1+y_n)^n} = 1
  \]
- The coupon bias is the difference between the par yield and the spot rate.
- Plotting \(y_t\) against \(t\) gives the yield curve.
The forward interest rate \( f_{t,k} \) is the interest on money borrowed from time \( T \) to time \( T+k \).

- Like the spot rate, but doesn't necessarily start at \( t=0 \).
- \( f_{0,k} = y_k \).
- Sometimes \( f_{t,k} \) means money borrowed from \( T \) to \( k \).
- The forward rate is the geometric mean of the spot rates:

\[
(1+y_e)^2 = (1+y_0)(1+f_{1,1})
\]
\[
(1+y_2)^3 = (1+y_0)^2 (1+f_{2,1}) = (1+y_1)(1+f_{1,1})(1+f_{2,1})
\]
\[
\vdots
\]
\[
(1+y_k)^k = (1+f_{0,1})(1+f_{1,1}) \cdots (1+f_{k-1,k})
\]
- i.e. the spot rate is the geometric mean of 1+y forward rates.

In the continuous case, we have the spot force of interest, and the forward force of interest (denoted by \( V_t \) and \( F_t \))

\[
(1+y_e)^t = e^{V_t}
\]
\[
(1+f_{t,k})^k = e^{kF_{t,k}}
\]

The instantaneous forward rate can be found by taking the limit:

\[
\lim_{k \to 0} F_{t,k}, \text{ leading to } P_t = \exp \left( - \int_0^t F_t \, dt \right)
\]

Expectations theory explains the yield curve in terms of expected future movements in interest rates:
- Interest rate expected to increase \( \Rightarrow \) long term bonds less attractive \( \Rightarrow \) yields increase to compensate.

Liquidity Preference theory:
- Short term bonds are more flexible and inherently less risky as a result. So long term bonds must offer higher yield.

Market segmentation theory suggests that there are different forces of supply and demand for bonds of different lengths:
- e.g. banks want short term, institutions want long term, etc.
Vulnerability to interest rate movements

- If $P$ is the present value of a series of cashflows, then the Macaulay duration / duration / discounted mean term is given by:

$$
\frac{dP}{P} = \frac{1}{P} \sum_{k=1}^{n} \frac{t_kc_{tk}}{(1+i)^{tk}}
$$

- the duration is the mean term of the cash flows weighted by their PVs. It has units of years.
- the duration measures sensitivity to interest rates.
- as the yield decreases, duration increases.

- The effective duration / modified duration / volatility of a cash flow is defined as:

$$
\tilde{d}(i) = \frac{1}{P} \frac{dP}{di} = \frac{1}{P} \sum_{k=1}^{n} \frac{t_kc_{tk}}{(1+i)^{tk-1}}
$$

- it is related to the Macaulay duration by $d_m(i) = (1+i) \tilde{d}(i)$
- if the yield decreases by a factor of $e$, price increases by a factor of $e^{\tilde{d}(i)}$.

- The convexity of a cash flow is the (positive) second derivative of price w.r.t. yield, per unit price.

$$
C(i) = \frac{1}{P} \frac{d^2P}{di^2} = \frac{1}{P} \sum_{k=1}^{n} \frac{t_k(t_k+i)c_{tk}}{(1+i)^{tk+2}}
$$

- higher positive convexity is good for the investor.
- convexity is related to variance: a more spread out cash flow is usually more convex.

- A fund has Redington immunization at $t_0$ subject to 3 conditions:
  1. $NPV(assets) = NPV(liabilities)$
  2. $d_m(assets) = d_m(liabilities)$
  3. $C(assets) > C(liabilities)$

- Condition 3 is equivalent to:

$$
\sum_{k=1}^{n} t_k^2a_{tk}v_{tk} \geq \sum_{k=1}^{n} t_k^2c_{tk}v_{tk}
$$
Stochastic interest rate problems

- If it is the interest rate from time \( t-1 \) to \( t \), the accumulated value of a unit investment at time \( n \) is:
  \[ s_n = (1+i_1)(1+i_2) \ldots (1+i_n) \]

- If it is a random variable with mean \( \mu \) and variance \( \sigma^2 \), and \( i_1, i_2, \ldots, i_n \) are independent:
  \[ E(s_n) = (1+\mu)^n \]
  \[ E(s_n^2) = [(1+\mu)^2 + \sigma^2]^n \]
  \[ \text{Var}(s_n) = [(1+\mu)^2 + \sigma^2]^n - (1+\mu)^{2n} \]

- If there are multiple investments, \( E(A_n) \) and \( E(A_n^2) \) can be calculated by noting that \( A_n = (1+i_n)(1+A_{n-1}) \).

- For a random variable \( Y \), if \( Y \sim \mathcal{N}(\mu, \sigma^2) \), then:
  \[ \ln Y \sim \text{Lognormal}(\mu, \sigma^2) \]  
  \( \text{not the mean/var of the lognormal dist!} \)

- If \( Y_i \sim \text{Lognormal}(\mu, \sigma^2) \), then \( \prod_{i=1}^{n} Y_i \sim \text{Lognormal}(n\mu, n\sigma^2) \), i.e. the product of lognormal distributions is a lognormal variable.

- Expectation of a lognormal dist: \( e^{\mu + \frac{1}{2}\sigma^2} \)
- Variance: \( e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \)

- Probability calculations will require the table of \( \Phi(x) \), usually with interpolation.
7. **Arbitrage**

- If investment $A$ costs $S_A$ and $B$ costs $S_B$, arbitrage exists in any of the following cases:
  - $\text{NPV}(A) = \text{NPV}(B)$ and $S_A \neq S_B$
  - $\text{NPV}(A) \neq \text{NPV}(B)$ and $S_A = S_B$
  - $S_A > S_B$ and $\text{NPV}(A) \leq \text{NPV}(B)$

- Realizing the arbitrage involves buying some of $A$ and shorting some $B$.

- A forward contract specifies that $A$ will buy asset $S$ from $B$ for price $K$, at time $T$.

- In the simplest case of a constant force of interest with no income, the contract can be priced by considering the NPV

$$S_0 = K e^{-\delta T}$$

- If there are fixed coupons, we can amend the above to include $CE^{-\delta t}$.

- However, it is different when the dividend yield is paid on the (unknown) price of the asset. By the no-arbitrage assumption, at time $T$ we require that $S \geq K$. Then we have:

$$S_0 e^{-\delta T} = K e^{-\delta T}$$

at an intermediate time.

- The value of a long contract can be calculated by finding $K_T - K_0$, where $T$ is the time of valuation, then discounting.

$$V_T = S_T - S_0 e^{-\delta T}$$